University of M'Sila,
Faculty of Mathematics and Computer Science / Department of Mathematics
Master of **Numerical and Mathematical Analysis S 2**
– ***Distributions Analyse de Fourier II***–

Final Exam May 2026

** We use the notations of the lesson.
** We take into account of the presentation of the copy.

Exercise 1. (8 pts.) (1) (Lesson.) Give the definition of :
– $\mathcal{S}(\mathbb{R})$, $\mathcal{S}'(\mathbb{R})$, $H^s(\mathbb{R})$. – The Fourier transform of a function. – The Fourier transform of a distribution. – The classification of partial differential equations of 2^{ed} order.

(2) (a) Find the two first integrals of the system

$$\frac{\mathrm{d}x}{y+1} = \frac{-\mathrm{d}y}{x+1} = \frac{-(y+1)\mathrm{d}u}{u(y-x)}. \qquad (E)$$

(b) Find the equation associated to (E) and its solution.

Exercise 2. (7 pts.) (1) Let $f(x)) = e^{-\pi x^2}$. Let $g(x) = e^{-\beta x^2}\cos(2\pi\beta x)$, $\beta > 0$. Calculate $\widehat{f}$. Calculate $\widehat{g}$. Calculate $\int_0^\infty x^2 e^{-2x^2}\cos(4\pi x)\mathrm{d}x$.

(2) Let $a \in \mathbb{R}$, $f_a(x) = \mathrm{e}^{2iax}$, $(i^2 = -1)$, and $k(x) = x\sin(2\pi x)$.

(a) Prove that $f_a, k \in \mathcal{S}'(\mathbb{R})$.

(b) Calculate $\widehat{f_a}$, $\widehat{k}$.

(c) Let $\gamma \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}\gamma \subset [-\frac{1}{2}, \frac{1}{2}]$. We put $G(x) = \gamma(x+3)$ and $a = \pi$. Calculate $(\mathcal{F}^{-1}G) * f_\pi$.

Exercise 3. (5 pts.) (1) Let the following equation

$$16\frac{\partial^2 u}{\partial x^2} - 8\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 4\sin x\,. \qquad (E)$$

(a) What is the nature of (E) ?
(b) We put $\xi = x + 4y$ and $\eta = 2y$, find the solution of (E).

(2) Find the solution of the equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1 + \left(\frac{y}{x}\right)^2 \qquad (x \in \mathbb{R}^*,\ y \in \mathbb{R})\,.$$

Scale :
Ex. 1 : 5 +[(0.5 + 2) + (0.25 + 0.25)]
Ex. 2 : (1+1+2) + (0.5+0.5) + (0.5+1)+0.5
Ex. 3 : (0.25 + 2) + 2.75

Exercise 1. (1) See the lesson.

(2) We put $x+1 = X$ and $y+1 = Y$, then $(E) \Leftrightarrow$

(a)

$$\underbrace{\frac{dX}{Y} = -\frac{dY}{X}}_{} = \frac{-Y\,du}{u(Y-X)} = \frac{Y\,dX}{Y^2} = \frac{-X\,dY}{X^2} = \frac{Y\,dX - X\,dY}{Y^2+X^2}$$

$$X\,dX + Y\,dY = 0$$

i.e. $\underline{X^2+Y^2 = c_1}$.

$$= \frac{\frac{1}{Y}dX - \frac{X}{Y^2}dY}{1+\left(\frac{X}{Y}\right)^2} = d\left(\operatorname{Arctg}\frac{X}{Y}\right)$$

$$\frac{-Y\,du}{u(Y-X)} = \frac{-u\,dY}{uX} = -\frac{u\,dY + Y\,du}{uY} = -\frac{d(uY)}{uY}$$

$$-\frac{d(uY)}{uY} = d\operatorname{Arctg}\frac{X}{Y} \Rightarrow \underline{\operatorname{Log}\left|\frac{uY}{c_2}\right| = \operatorname{Arctg}\frac{X}{Y}}$$

Conclusion:

$$\begin{cases} (x+1)^2 + (y+1)^2 = c_1 \\ u(y+1) = c_2\, e^{\operatorname{Arctg}\frac{x+1}{y+1}} \end{cases}$$

(b)

- $(y+1)\dfrac{\partial u}{\partial x} - (x+1)\dfrac{\partial u}{\partial y} = -\dfrac{u(y-x)}{y+1}$
- Solution we write $c_1 = \varphi(c_2)$ with $\varphi : \mathbb{R} \to \mathbb{C}$.

Exercise 2. (1) • $\hat{f}(\xi) = e^{-\pi\xi^2}$ See the lesson since $f(x) = e^{-\pi x^2}$.

• $g \in L_1(\mathbb{R})$, then

$$\hat{g}(\xi) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-2i\pi x\xi - \beta x^2}\left(e^{2\pi\beta i x} + e^{-2\pi\beta i x}\right)dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-2i\pi x(\xi-\beta)}\cdot e^{-\beta x^2}\,dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{-2i\pi x(\xi+\beta)}\cdot e^{-\beta x^2}\,dx$$

Since $\cos\alpha = \frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right)$ and $e^{-\beta x^2} = f\left(\sqrt{\frac{\beta}{\pi}}\,x\right)$

$$\hat{g}(\xi) = \frac{1}{2}\mathcal{F}\left(f\left(\sqrt{\tfrac{\beta}{\pi}}\,x\right)\right)(\xi-\beta) + \frac{1}{2}\mathcal{F}\left(f\left(\sqrt{\tfrac{\beta}{\pi}}\,x\right)\right)(\xi+\beta)$$

$\boxed{2|4}$

$\mathcal{F}(f(\lambda x))(\xi) = \lambda^{-1}\mathcal{F}f(\xi/\lambda)$ $\forall \lambda > 0$, then

$$\hat{g}(\xi) = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}\left[\hat{f}\left(\sqrt{\tfrac{\pi}{\beta}}(\xi-\beta)\right) + \hat{f}\left(\sqrt{\tfrac{\pi}{\beta}}(\xi+\beta)\right)\right]$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{\beta}}\left[e^{-\frac{\pi^2}{\beta}(\xi-\beta)^2} + e^{-\frac{\pi^2}{\beta}(\xi+\beta)^2}\right].$$

- We use the formula

$$(\hat{g}(\xi))'' = (-2i\pi)^2\,\widehat{x^2 g(x)}(\xi) = -4\pi^2\,\widehat{x^2 g(x)}(\xi)$$

with $\xi = 0$ and $\beta = 2$.

$$(\hat{g}(\xi))'' = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}\left[\left(-2\frac{\pi^2}{\beta} + \frac{4\pi^4}{\beta^2}(\xi-\beta)^2\right)e^{-\frac{\pi^2}{\beta}(\xi-\beta)^2} + \left(-2\frac{\pi^2}{\beta} + 4\frac{\pi^4}{\beta^2}(\xi+\beta)^2\right)e^{-\frac{\pi^2}{\beta}(\xi+\beta)^2}\right]$$

Now

$$\hat{g}''(0) = -4\pi^2\int_{-\infty}^{\infty} x^2 e^{-2x^2}\cos(4\pi x)\,dx \qquad (\beta = 2)$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{2}}\,e^{-\pi^2\beta}\left[4\pi^4 + 4\pi^4 - 2\pi^2\right] = \frac{\pi^2\sqrt{\pi}}{\sqrt{2}}\,e^{-\pi^2\beta}(4\pi^2-1)$$

$$\boxed{\int_0^{\infty} x^2 e^{-2x^2}\cos(4\pi x)\,dx = \frac{1}{8}(1-4\pi^2)\sqrt{\frac{\pi}{2}}\,e^{-2\pi^2}.}$$

(2) $\varphi \in S(\mathbb{R})$

(a) $|\langle f_a|\varphi\rangle| = \left|\int_{\mathbb{R}} e^{2iax}\varphi(x)\,dx\right| \le \|\varphi\|_{L^1(\mathbb{R})} < +\infty.$

Similar for k.

(b) $\langle \hat{f}_a|\varphi\rangle := \langle f_a|\hat{\varphi}\rangle = \int_{\mathbb{R}} e^{2iax}\hat{\varphi}(x)\,dx = \int_{\mathbb{R}} e^{2ia\xi}\hat{\varphi}(\xi)\,d\xi$

$= \varphi\left(\frac{a}{\pi}\right) = \langle \delta_{a/\pi}|\varphi\rangle \Rightarrow \hat{f}_a(\xi) = \delta_{\xi = \frac{a}{\pi}}.$

- We put $h(x) = \sin 2\pi x$,

$$\mathcal{F}(xh)(\xi) = \frac{-1}{2i\pi}(\hat{h}(\xi))' \quad \text{since } k(x) = x h(x)$$

$$h(x) = \frac{1}{2i}\left(e^{2i\pi x} - e^{-2i\pi x}\right) = \frac{1}{2i}\left(f_{\pi}(x) - f_{-\pi}(x)\right)$$

$$\Rightarrow \hat{h}(\xi) = \frac{1}{2i}\left(\delta_{\xi=1} - \delta_{\xi=-1}\right)$$

$$\Rightarrow \mathcal{F}(xh(x))(\xi) = \frac{-1}{(2i)^2\pi}\left(\delta'_{\xi=1} - (-1)\delta'_{\xi=-1}\right)$$

$$\Rightarrow \hat{k}(\xi) = \frac{1}{4\pi}\left(\delta'_{\xi=1} + \delta'_{\xi=-1}\right).$$

(c) We have $\operatorname{supp} G \subset \left[-\frac{7}{2}, -\frac{5}{2}\right]$

$$\mathcal{F}\left(\mathcal{F}^{-1}G * f_{\pi}\right)(\xi) = G(\xi)\hat{f}_{\pi}(\xi)$$

$$= G(\xi)\delta_{\xi=1} = G(1)\delta_{\xi=1} = 0 \quad \text{Since } 1 \notin \operatorname{supp} G$$

$$\Rightarrow \mathcal{F}^{-1}G * f_{\pi} = 0.$$

Exercise 3. (1)(a) Parabolic.

(b) We put $u(x,y) = v(\xi,\eta)$, $\xi = x + 4y$, $\eta = 2y$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi} \qquad \frac{\partial u}{\partial y} = 4\frac{\partial v}{\partial \xi} + 2\frac{\partial v}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial \xi^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = 4\frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \eta}$$

$$\frac{\partial^2 u}{\partial y^2} = 16\frac{\partial^2 v}{\partial \xi^2} + 16\frac{\partial^2 u}{\partial \xi \partial \eta} + 4\frac{\partial^2 v}{\partial \eta^2}$$

$$(E) \Leftrightarrow \frac{\partial^2 v}{\partial \eta^2} = \sin x = \sin(\xi - 2\eta).$$

$$\frac{\partial v}{\partial \eta} = -\frac{1}{2}\cos(\xi - 2\eta) + F(\xi)$$

$$v = \frac{1}{4}\sin(\xi - 2\eta) + F(\xi)\eta + G(\xi)$$

$$\Rightarrow \boxed{u(x,y) = \frac{1}{4}\sin x + 2y\,F(x+4y) + G(x+4y).}$$ $F, G : \mathbb{R} \to \mathbb{C}$

We put $x = r\cos\theta$, $y = r\sin\theta$ and $u(x,y) = W(r,\theta)$.

$$\frac{\partial W}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \quad \Big)\cos\theta \qquad \Big)\sin\theta$$

$$\frac{\partial W}{\partial \theta} = -\frac{\partial u}{\partial x} r\sin\theta + \frac{\partial u}{\partial y} r\cos\theta \quad \Big)-\frac{1}{r}\sin\theta \qquad \frac{1}{r}\cos\theta$$

$$\Rightarrow \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial W}{\partial r}\cos\theta - \dfrac{\partial W}{\partial \theta}\dfrac{\sin\theta}{r} \\ \dfrac{\partial u}{\partial y} = \dfrac{\partial W}{\partial r}\sin\theta + \dfrac{\partial W}{\partial \theta}\dfrac{\cos\theta}{r} \end{cases}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = r\frac{\partial W}{\partial r} = 1 + \left(\frac{y}{x}\right)^2 = 1 + (tg\,\theta)^2$$

$$\frac{\partial W}{\partial r} = \left(1 + (tg\,\theta)^2\right)\frac{1}{r}$$

$$W = \left(1 + (tg\,\theta)^2\right)\operatorname{Log} r + \lambda(\theta)$$

$$\boxed{u(x,y) = \left(1 + \left(\frac{y}{x}\right)^2\right)\operatorname{Log}(x^2+y^2) + \lambda\left(\operatorname{Arctg}\frac{y}{x}\right)}$$

where $\lambda : \mathbb{R} \longrightarrow \mathbb{C}$.