University of M'Sila,
Faculty of Mathematics and Computer Science / Department of Mathematics
Master of Numerical and Mathematical Analysis S 2
– Distributions Analyse de Fourier II–

Final Exam May 2025

** We use the notations of the lesson.
** We take into account of the presentation of the copy.

Exercise 1. (8 pts.) (1) (Lesson.) Give the definition of :
– $\mathcal{S}(\mathbb{R})$. – $\mathcal{S}'(\mathbb{R})$. – $H^s(\mathbb{R})$. – The Fourier transform of a distribution. – The classification of partial differential equations of 2^{ed} order.

(2) (a) Find the two first integrals of the system

$$\frac{dx}{-y} = \frac{dy}{x+1} = du\,. \qquad (G)$$

(b) Find the equation associated to (G) and its solution.

Exercise 2. (7 pts.) (1) Let $a, b \in \mathbb{R}$. Calculate $\delta_a * \delta_b$.
(2) Let $a \in \mathbb{R}$, $f_a(x) = e^{iax}$, $(i^2 = -1)$, $g(x) = \cos(2\pi x)$ and $h(x) = (\cos(2\pi x))^2$.

(a) Prove that $f_a, g \in \mathcal{S}'(\mathbb{R})$.
(b) Calculate $\widehat{f_a}$, $\widehat{g}$, $\widehat{h}$.

(3) Let $\gamma \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}\gamma \subset [-1, 1]$. We put $a = 4\pi$. Calculate $(\mathcal{F}^{-1}\gamma) * f_{4\pi}$.

Exercise 3. (5 pts.) (1) Let the following equation

$$\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} = e^y\,. \qquad (E)$$

(1-a) What is the nature of (E) ?
(1-b) We put $\xi = 2x + y$ and $\eta = x$, find the solution of (E).

(2) Let the following equation

$$x^2\frac{\partial^2 u}{\partial x^2} - y2\frac{\partial^2 u}{\partial y^2} = 0 \qquad (x > 0,\, y > 0)\,. \qquad (F)$$

(2-a) What is the nature of (F) on $\mathbb{R}^+ \times \mathbb{R}^+$?
(2-b) We put $\xi = y/x$ and $\eta = xy$, find the solution of (F) on $\mathbb{R}^+ \times \mathbb{R}^+$.

Scale :
Ex. 1 : 5 +[(0.5 + 2) + (0.25 + 0.25)]
Ex. 2 : 1 + [(1+1) + (1+1+1)] + 1
Ex. 3 : (0.25 + 1.5) + (0.25 + 3)

CORRECTION

M1 AMN Distributions Analyse de Fourier 1)

5 **Exercise 1.** (1) See the lesson.

(2) (a) We put $x+1 = X$, then (G) $\Longleftrightarrow$

$$\underbrace{\frac{dX}{-y} = \frac{dy}{X}}_{\swarrow} = du = \underbrace{\frac{-y\,dX}{y^2} = \frac{X\,dy}{X^2} = \frac{-y\,dX + X\,dy}{y^2+X^2}}_{\text{see the lesson}}$$

$$X\,dX + y\,dy = 0$$

i.e. $X^2 + y^2 = c_1$

$$= \frac{-\frac{y}{X^2}dX + \frac{1}{X}dy}{\left(\frac{y}{X}\right)^2 + 1} = d\,\mathrm{Arctg}\,\frac{y}{X}$$

$$du = d\,\mathrm{Arctg}\,\frac{y}{X}$$

$$u = \mathrm{Arctg}\,\frac{y}{X} + c_2$$

0,5

2,5

$$\begin{cases} (x+1)^2 + y^2 = c_1 \\ u = \mathrm{Arctg}\,\dfrac{y}{x+1} + c_2 . \end{cases}$$

0,25 (b) • $-y\dfrac{\partial u}{\partial x} + (x+1)\dfrac{\partial u}{\partial y} = 1$.

• $c_2 = \varphi(c_1)$, i.e.

0,25 $$u = \varphi\left((x+1)^2 + y^2\right) + \mathrm{Arctg}\,\frac{y}{x+1}$$

where $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$.

1 **Exercise 2** (1) $\delta_a * \delta_b = \delta_{a+b}$

indeed $\varphi \in \mathcal{D}(\mathbb{R})$:

$$\langle \delta_a * \delta_b | \varphi \rangle = \langle \underset{x}{\delta_a}, \langle \underset{y}{\delta_b} | \varphi(x+y) \rangle \rangle$$
$$= \langle \underset{x}{\delta_a} | \varphi(x+b) \rangle = \varphi(a+b) = \langle \delta_{a+b} | \varphi \rangle.$$

1 (2) (a) $\varphi \in S(\mathbb{R})$,
$$|\langle f_a | \varphi \rangle| = \left| \int_{\mathbb{R}} e^{iax} \varphi(x)\, dx \right| \leq \int_{\mathbb{R}} |\varphi(x)|\, dx = \|\varphi\|_{L^1(\mathbb{R})}$$

1 (i.e. $f \in S'(\mathbb{R})$) similarly for g.

1 (b) • $\hat{f}_a(\xi) = \delta_{\xi = a/2\pi}$, indeed, $\varphi \in S(\mathbb{R})$ we have

$$\langle \hat{f}_a | \varphi \rangle := \langle f_a, \hat{\varphi} \rangle = \int_{\mathbb{R}} e^{iax} \hat{\varphi}(x)\, dx = \int_{\mathbb{R}} e^{ia\xi} \hat{\varphi}(\xi)\, d\xi$$
$$= \varphi(a/2\pi) = \langle \delta_{a/2\pi} | \varphi \rangle.$$

1 • $g(x) = \frac{1}{2}\left(e^{2\pi i x} + e^{-2i\pi x}\right) = \frac{1}{2}\left(f_{2\pi}(x) + f_{-2\pi}(x)\right)$

then $\hat{g}(\xi) = \frac{1}{2}(\delta_1 + \delta_{-1})$.

1 • $h(x) = (g(x))^2$ i.e. $\hat{h}(\xi) = \hat{g} * \hat{g}(\xi)$

$$\hat{h}(\xi) = \frac{1}{4}(\delta_1 + \delta_{-1}) * (\delta_1 + \delta_{-1})$$
$$= \frac{1}{4}(\delta_2 + 2\delta_0 + \delta_{-2}). \quad \text{since } \delta_a * \delta_b = \delta_{a+b}$$

1 (3) $\mathcal{F}\left(\mathcal{F}^{-1}\gamma * f_{4\pi}\right)(\xi) = \gamma(\xi) \hat{f}_{4\pi}(\xi)$

$= \gamma(\xi)\, \delta_{2=\xi}$ but $\gamma(2) = 0$

$= \gamma(2)\, \delta_{\xi=2} = 0$ then $\mathcal{F}^{-1}\gamma * f_{4\pi} \equiv 0$.

0,25 Exercise 3. (1) (1-a) Parabolic.

(1-b) $\frac{\partial u}{\partial x} = 2\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}$ $\quad (u(x,y) = v(\xi,\eta))$

$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial \xi}$ $\quad \begin{cases} \xi = 2x+y \\ \eta = x \end{cases}$

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial^2 v}{\partial \xi^2} + 4\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = +2\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta \partial \xi}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial \xi^2}$$

$$\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial \eta^2} = e^{y} = e^{\xi - 2\eta}$$

$$(E) \Leftrightarrow \frac{\partial^2 v}{\partial \eta^2} = e^{\xi - 2\eta}$$

- $\frac{\partial v}{\partial \eta} = -\frac{1}{2}e^{\xi-2\eta} + F(\xi)$ $\quad 1^{st}$ integration / η
- $v = +\frac{1}{4}e^{\xi-2\eta} + \eta F(\xi) + G(\xi)$ $\quad 2^{nd}$ integration / η

1,5 $$\boxed{u = \frac{1}{4}e^{y} + x F(2x+y) + G(2x+y)}$$

0,25 (2) (2-a) Hyperbolic.

(2-b) We put $u(x,y) = \omega(\xi,\eta)$ $\quad \xi = y/x, \; \eta = xy$

$\frac{\partial u}{\partial x} = \frac{\partial \omega}{\partial \xi}\left(-\frac{y}{x^2}\right) + \frac{\partial \omega}{\partial \eta} y$, $\quad \frac{\partial u}{\partial y} = \frac{\partial \omega}{\partial \xi}\left(\frac{1}{x}\right) + \frac{\partial \omega}{\partial \eta} x$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 \omega}{\partial \xi^2}\left(\frac{y}{x^2}\right)^2 + \frac{\partial^2 u}{\partial \eta^2} y^2 - \frac{2y^2}{x^2}\frac{\partial^2 \omega}{\partial \xi \partial \eta} + \frac{2y}{x^3}\frac{\partial \omega}{\partial \xi}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 \omega}{\partial \xi^2}\left(\frac{1}{x^2}\right) + \frac{\partial^2 \omega}{\partial \eta^2} x^2 + 2\frac{\partial^2 \omega}{\partial \xi \partial \eta}$$

$$(F) \Leftrightarrow \frac{\partial^2 \omega}{\partial \xi \partial \eta} = \frac{1}{2\eta}\frac{\partial \omega}{\partial \xi}$$

1,5 Integrating / ξ

$$\frac{\partial w}{\partial \eta} = \frac{1}{2\eta} w + F(\eta)$$

this equation has the following form

$$y' = \frac{1}{2t} y + F(t) \quad \text{---- (B)}$$

a linear equation.

- If we limit ourself to $t \geq 0$
  (B) has $y = c\sqrt{t} + g(t)$ a solution
  g is obtained from F with the standard method.

  Note that $t \mapsto \eta$
  $c \mapsto c(\xi) =: \varphi(\xi)$
  $y \mapsto w(\xi, \eta)$

  then $w(\xi, \eta) = \varphi(\xi)\sqrt{\eta} + g(\eta)$.

- On $\mathbb{R}^+ \times \mathbb{R}^+$ i.e $xy = \eta > 0$
  we obtain the solution:

$$u(x,y) = \sqrt{xy}\,\varphi\left(\frac{y}{x}\right) + g(xy)$$

1,5 where $\varphi, g : \mathbb{R} \longrightarrow \mathbb{C}$.

4 | 4