University of M'Sila,
Faculty of Mathematics and Computer Science / Department of Mathematics
Master of Numerical and Mathematical Analysis / S 1
Distributions Analyse de Fourier 1

Final Exam, January 2026

*We use the notations of the lesson.

Exercise 1 (lesson). Give the definition of :
(1)– a regular distribution, – a singular distribution,
(2)– the $L^1_{\mathrm{loc}}(\mathbb{R})$ space,
(3)– the distributions δ_0, $\mathrm{pv}\frac{1}{x}$,
(4)– the derivation in $\mathcal{D}'(\mathbb{R})$, – the convolution of the functions,
(5)– the convergence in $\mathcal{D}'(\mathbb{R})$.

Exercise 2. (Independent questions)
(1) Let $\langle T,\varphi\rangle = \lim\limits_{\varepsilon\to 0}\left(\int_{-\infty}^{-\varepsilon}\frac{\varphi(x)}{x^2}\,\mathrm{d}x + \int_{\varepsilon}^{\infty}\frac{\varphi(x)}{x^2}\,\mathrm{d}x - 2\frac{\varphi(0)}{\varepsilon}\right)$, $\forall\varphi\in\mathcal{D}(\mathbb{R})$. Prove that $T\in\mathcal{D}'(\mathbb{R})$.

(2) Define the Heaviside function $H(x)$. We put $T = xH(3+x)$.
Calculate $H'(x)$ and T' in $\mathcal{D}'(\mathbb{R})$.

(3) Let $\langle T,\varphi\rangle = \int_0^{\infty}\varphi'(x)\cos x\,\mathrm{d}x$, $\forall\varphi\in\mathcal{D}(\mathbb{R})$.
Prove that $T\in\mathcal{D}'(\mathbb{R})$. Calculate T, T', T'' in $\mathcal{D}'(\mathbb{R})$.

(4) Let $g\in L^2(\mathbb{R})$. Let $\langle U,\varphi\rangle = \int_{-\infty}^{\infty} g(x)\,(\mathrm{e}^{-x^2} - 3\cos x)\,\varphi(x)\,\mathrm{d}x$, $\forall\varphi\in\mathcal{D}(\mathbb{R})$.
Prove that $U\in\mathcal{D}'(\mathbb{R})$. Application with $g(x) = |x|^{\frac{1}{2}}\,\mathrm{e}^{-\frac{x^2}{2}}$.

(5) Let $m\in\mathbb{N}^*$. Let $k\in C^{\infty}(\mathbb{R})$ and $T = k(x)H(x)$.
(a) Prove that $T\in\mathcal{D}'(\mathbb{R})$. Calculate $T^{(m)}$ in $\mathcal{D}'(\mathbb{R})$.
(b) We put $k(x) = \mathrm{e}^x$. Calculate $S = T^{(m)} - T$ in $\mathcal{D}'(\mathbb{R})$.

(6) Let $h\in C^{\infty}(\mathbb{R})$. Calculate $S_1 = x\delta_0'$ and $S_2 = (x\delta_0') * (h(x)\delta_0)$ in $\mathcal{D}'(\mathbb{R})$.

Exercise 3. Let $V_j := j\mathrm{e}^{-jx}H(x)$, $j\in\mathbb{N}^*$.
Prove that $V_j\in\mathcal{D}'(\mathbb{R})$. Calculate $\lim_{j\to\infty} V_j$ in $\mathcal{D}'(\mathbb{R})$.

Scale :
Ex. 1 **(5 pts.)** : (0.5+0.5) + 1 + (0.5+0.5) + (0.5+0.5) + 1
Ex. 2 **(12 pts.)** : (0.25 linear, 1.75) + (0.5+0.5+1) + ((0.25+0.75)+0.5+0.25+0.25) + (1+1) + (0.25+1+0.75) + (1+1)
Ex. 3 **(3 pts.)** : 0.5 + 2.5

CORRECTION

Exercise 1

See the lesson.

Exercise 2

(1) $\varphi(x) = \varphi(0) + x\varphi'(0) + x^2\psi(x)$; $|\psi(x)| \leq \sup|\varphi''|$ $\boxed{0,5}$

$\varphi \in \mathcal{D}(\mathbb{R})$, $\exists M>0$, $\operatorname{supp}\varphi \subseteq [-M,M]$.

$$\int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x^2}dx + \int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x^2}dx = \int_{-M}^{-\varepsilon} \frac{\varphi(x)}{x^2}dx + \int_{\varepsilon}^{M} \frac{\varphi(x)}{x^2}dx$$

$$= -\int_{M}^{\varepsilon} \frac{\varphi(-x)}{(-x)^2}dx + \int_{\varepsilon}^{M} \frac{\varphi(x)}{x^2}dx = \int_{\varepsilon}^{M} \frac{\varphi(x)+\varphi(-x)}{x^2}dx$$

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$$\varphi(x) + \varphi(-x) = 2\varphi(0) + x^2(\psi(x)+\psi(-x)) = 2\varphi(0) + x^2\tilde{\psi}(x)$$

$$\int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x^2}dx + \int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x^2}dx = 2\varphi(0)\left(-\frac{1}{M}+\frac{1}{\varepsilon}\right) + \int_{\varepsilon}^{M}\tilde{\psi}(x)\,dx$$

$|\tilde{\psi}(x)| \leq 2\sup|\varphi''|$

$$\left|\int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x^2}dx + \int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x^2}dx - 2\frac{\varphi(0)}{\varepsilon}\right| \leq \frac{2|\varphi(0)|}{M} + 2\sup|\varphi''|(M-\varepsilon)$$

$$\lim_{\varepsilon\downarrow 0}\left|\int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x^2}dx + \int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x^2}dx - 2\frac{\varphi(0)}{\varepsilon}\right| \leq \frac{2}{M}|\varphi(0)| + 2M\sup|\varphi''|$$ $\boxed{1}$

then $|\langle T,\varphi\rangle| \leq \left(\frac{2}{M}+M\right)\|\varphi\|_{\mathcal{D}}$,

- T is linear (easy). $\boxed{0,25}$
- and T is continuous $\boxed{0,25}$

(2)
- $H(x) = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x<0 \end{cases}$ $\boxed{0,5}$
- $\langle H',\varphi\rangle = -\langle H,\varphi'\rangle$ $\boxed{0,25}$ $= -\int_0^{\infty}\varphi'(x)\,dx = \varphi(0) = \langle\delta,\varphi\rangle$

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$\Rightarrow$ $\underline{H' = \delta}$, in $\mathcal{D}'(\mathbb{R})$ $\boxed{0,25}$

- $T' =$ $\boxed{0,5}$ $H(3+x) + xH'(3+x) = H(3+x) + x\delta_{3+x=0}$ $\boxed{0,25}$

$= H(3+x) - 3\delta_{x=-3}$ $\boxed{0,25}$ in $\mathcal{D}'(\mathbb{R})$.

(3). T is linear (easy). $\boxed{0{,}25}$

- $|\langle T,\varphi\rangle| \leq \int_0^M |\varphi'(x)||\cos x|\,dx \leq \int_0^M |\varphi'(x)|\,dx$ $\boxed{0{,}25}$
  $\leq M \sup|\varphi'| \leq M\,\|\varphi\|_{\mathcal{D}}$ $\boxed{0{,}5}$. T is continuous

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- $\langle T,\varphi\rangle = \int_0^\infty \varphi'(x)\cos x\,dx = \int_{-\infty}^\infty H(x)\cos x\,.\,\varphi'(x)\,dx$
  $= \langle \cos x.H, \varphi'\rangle = -\langle -\sin x\, H + \cos x\,\delta, \varphi\rangle$ $\boxed{0{,}5}$
  $T = \sin x\, H - \cos x\,\delta = \underline{\sin x\, H - \delta_{x=0}}$
  (we know $\cos x\,\delta = (\cos 0)\,\delta = \delta$)
- $T' = (\sin x\, H - \delta)' = \cos x\, H + \sin x\,\delta - \delta'$
  $= \underline{\cos x\, H - \delta'}$, $\boxed{0{,}25}$ (since $\sin x\,\delta = (\sin 0)\,\delta = 0$)
- $T'' = (\cos x\, H - \delta')' = -\sin x\, H + \cos x\,\delta - \delta''$
  $= \underline{-\sin x\, H + \delta - \delta''}$ $\boxed{0{,}25}$ in $\mathcal{D}'(\mathbb{R})$.

(4) • U is linear (easy). $\boxed{0{,}25}$

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- $|\langle U,\varphi\rangle| \leq \int_{-\infty}^{\infty} |g(x)| \left[e^{-x^2} + 3|\cos x|\right] |\varphi(x)|\,dx$ $\boxed{0{,}5}$
  $\leq 4\int_{-\infty}^{\infty} |g(x)||\varphi(x)|\,dx \leq 4\left(\int_{-\infty}^{\infty} |g(x)|^2 dx\right)^{1/2} \left(\int_{-\infty}^{\infty} |\varphi(x)|^2 dx\right)^{1/2}$ $\boxed{0{,}25}$
  $\leq 4\|g\|_{L^2} \left(\int_{-M}^{M} |\varphi|^2\right)^{1/2} \leq 4\sqrt{2M}\,\|g\|_{L^2}\,\|\varphi\|_\infty$. U is continuous.
- To prove $g \in L^2(\mathbb{R})$:
  $\int_{-\infty}^{\infty} |g(x)|^2 dx = \int_{-\infty}^{\infty} |x|\,e^{-x^2} dx = 2\int_0^\infty x\,e^{-x^2} dx = -e^{-x^2}\Big|_0^\infty$ $\boxed{1}$
  $= 1$ then $\|g\|_{L^2} = 1$.

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(5) (a). $H \in \mathcal{D}'(\mathbb{R})$ and $k \in C^\infty$

$\Rightarrow k(x) H \in \mathcal{D}'$ (see the lesson) $\boxed{0,25}$

- $T' = k'(x) H + k(x)\delta = k'(x)H + k(0)\delta$ $\boxed{0,25}$ $(k(x)\delta = k(0)\delta)$

$T'' = k''(x) H + k'(x)\delta + k(0)\delta' = k''(x) H + k'(0)\delta + k(0)\delta'$ $\boxed{0,25}$

...

2 $T^{(m)} = k^{(m)}(x) H + k^{(m-1)}(0)\delta + k^{(m-2)}(0)\delta' + \dots + k(0)\delta^{(m-1)}$ $\boxed{0,5}$

" to prove this by induction on m" (easy).

- $k(x) = e^x$ then $k(0) = k'(0) = \dots = k^{(m-1)}(0) = 1$ $\boxed{0,25}$

and $k^{(m)}(x) H = e^x H = k(x) H = T$

$\Rightarrow S = T^{(m)} - T = \delta + \delta' + \delta'' + \dots + \delta^{(m-1)}$ $\boxed{0,5}$.

(6). $\langle S_1 | \varphi\rangle = \langle x\delta' | \varphi\rangle = \langle \delta', x\varphi\rangle = -(x\varphi)'(0)$ $\boxed{0,5}$

$= -\varphi(x) - x\varphi'(x)\Big|_{x=0} = -\varphi(0) = \langle -\delta | \varphi\rangle$ $\boxed{0,25}$

2 $\underline{S_1 = -\delta}$. $\boxed{0,25}$

- $S_2 = (x\delta') * (h(x)\delta) = -\delta * (h(x)\delta) = -h(x)\delta$ $\boxed{0,25}$ $\boxed{0,25}$

$= \underline{-h(0)\delta}$ $\boxed{0,5}$

Exercice 3 . $V_j(x) = j e^{-jx}$ if $x > 0$, $V_j(x) = 0$ if $x < 0 \Rightarrow V_j \in L^1_{Loc}(\mathbb{R})$

V_j is linear (easy). then $\underline{V_j \in \mathcal{D}'(\mathbb{R})}$. $\boxed{0,5}$

- $\langle V_j | \varphi\rangle = j\int_0^{\infty} e^{-jx}\varphi(x)\,dx = \int_0^\infty e^{-y}\varphi\left(\frac{y}{j}\right) dy$ $\boxed{0,5}$

$= \int_0^\infty e^{-y}\left(\varphi(0) + \frac{y}{j}\varphi'\left(\theta\frac{y}{j}\right)\right) dy = \varphi(0)\underbrace{\int_0^\infty e^{-y} dy}_{=1} + \frac{1}{j}\int_0^\infty y e^{-y}\varphi'\left(\theta\frac{y}{j}\right)$

$\frac{1}{j}\left|\int_0^\infty y e^{-y}\varphi'\left(\theta\frac{y}{j}\right) dy\right| \le \frac{1}{j}\sup|\varphi'|\int_0^\infty y e^{-y} dy = \frac{1}{j}\sup|\varphi'| \longrightarrow 0$ when $j \to +\infty$

$\Rightarrow \langle V_j, \varphi\rangle \to \varphi(0)$ i.e. $\underline{\lim V_j = \delta}$ in $\mathcal{D}'$.