English 1
Basic Mathematical Vocabulary

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Introduction

Mathematics occupies a central place in the advancement of science, technology, and modern problem-solving. Its importance lies not only in its ability to describe natural phenomena with precision, but also in the logical thinking, abstraction, and analytical skills it develops.

Learning and teaching mathematics in English has become increasingly valuable in academic and professional contexts. English is the dominant language of scientific communication, international collaboration, and research publication. Mastering mathematical terminology and explanations in English allows students to access global resources, follow advanced literature, present their work clearly, and participate effectively in international conferences and collaborations.

The main purpose of this handout is to help learners make a smooth transition toward **learning and teaching mathematics in English** by developing essential mathematical vocabulary, improving reading and interpretation skills, and providing clear rules and illustrative examples for the correct use of common symbols, expressions, and operations.

This handout is designed for first-year Master's students in Mathematical and Numerical Analysis, but it can also serve as a helpful reference for students in related programs, as well as for researchers and teachers who wish to learn, understand, and teach mathematical concepts confidently in English.

The handout contains the following components:

Chapter 1: "Basic Mathematical Vocabulary,"
presents the essential words and expressions used in basic mathematics. It explains how to read numbers, decimals, and fractions, introduces the main sets of numbers, and reviews factors and multiples. It also provides simple vocabulary related to sequences, series, logic, and sets. Its goal is to help students use clear and correct mathematical English.

Chapter 2: "Reading Math Expressions,"**
helps students understand how to read and say common mathematical expressions in English. It covers the four basic operations, comparing numbers, and important functions such as powers, roots, logarithms, exponentials, and trigonometric functions. The chapter also explains how to read common operators, including limits, derivatives, partial derivatives, and integrals. Its purpose is to make mathematical expressions clear and easy to read in English.

Further parts will be added in the future to deepen and extend other advanced topics, ensuring that the handout remains a growing and comprehensive resource for learning and teaching mathematics in English.

Chapter 1

Basic Mathematical Vocabulary

Mathematics is a vast and limitless field that covers countless concepts, ideas, and techniques. In this chapter, we present only a few essential sections that introduce the basic vocabulary used in mathematics. These topics are just a small selection compared to the unlimited number of subjects found in the world of mathematics. The aim is to provide students with the fundamental words and expressions needed to understand, describe, and communicate mathematical ideas with confidence.

1.1 Numbers

Numbers are symbols used to represent quantities, measure values, or show order. Digits are the basic symbols used to write numbers. There are 10 digits in the decimal (base-10) system:

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9$$

Examples:

- 4 is a single digit.
- The number 356 is made up of the digits 3, 5, and 6, where 3 is the hundreds digit, 5 is the tens digit, and 6 is the units digit.

1.1.1 Reading Numbers

In mathematics and everyday life, there are two main types of numbers: **cardinal** and **ordinal** numbers. Reading numbers varies depending on their classification.

Cardinal Numbers

Cardinal numbers are the names of numbers. They are used for counting.

Example 1.

- *The matrix A has 9 elements.*
- *There are 2 roots of the quadratic equation.*
- *0 is called nought, zero, or oh.*

The following table shows the spelling of some cardinal numbers.

1	one	11	eleven	21	twenty-one
2	two	12	twelve	29	twenty-nine
3	three	13	thirteen	30	thirty
4	four	14	fourteen	40th	forty
5	five	15	fifteen	50	fifty
6	six	16	sixteen	60	sixty
7	seven	17	seventeen	70	seventy
8	eight	18	eighteen	90	ninety
9	nine	19	nineteen	100	one hundred
10	ten	20	twenty	101	one hundred and one

200	two hundred	10 000	ten thousand
341	three hundred and forty-one	100 000	a hundred thousand
1 000	one thousand	1 000 000	one million
1 200	one thousand two hundred	1 000 000 000	one billion
1 125	a thousand one hundred and twenty-five	1 000 000 000 000	one trillion

Remark:

- The words *hundred*, *thousand*, etc. do not take an *s* after a number (e.g., 500 is *five hundred*, not *five hundreds*).
- In British English, the word *and* is used after *hundred*, while in American English it is usually omitted.

Ordinal Numbers

Ordinal numbers indicate position or order. They are also used when speaking about the day of the month.

Examples:

- The 5th iteration of Newton's method gives the best approximation.
- The 2nd derivative of the function $f(x)$ is $f''(x)$.
- The zeroth law of thermodynamics.

Remarks:

- Ordinal numbers usually end with *th*, except for *first*, *second*, and *third*.
- Some irregular forms: *fifth* (not fiveth), *ninth* (not nineth), *twelfth* (not twelveth).
- Add -st, -nd, -rd, or -th to digits when writing ordinals (1st, 2nd, 3rd, 4th, ...).

The following table shows the spelling of some ordinal numbers.

1st	first	21st	twenty-first
2nd	second	75th	seventy-fifth
3rd	third	100th	one hundredth
5th	fifth	101st	one hundred and first
12th	twelfth	1 000th	one thousandth

Practice

1. Write the spelling of the following numbers

   22, 103, 10 540, 608

2. Fill in the blanks with cardinal numbers

   - The equation has __________ unknowns.
   - The matrix contains __________ rows and __________ columns.
   - There are __________ digits in the decimal system.
   - The vector has __________ components.

3. Fill in the blanks with ordinal numbers

   - The __________ derivative of $f(x)$ represents velocity.
   - The __________ column of the matrix contains boundary values (9).
   - The __________ step in solving the problem is to simplify the equation (3).
   - The __________ term of the series is zero (10).
   - The __________ iteration gives the best approximation (15).

1.1.2 Reading Numbers with Decimals

The dot (.) is called a *decimal point*. Read the whole number part normally, then say "point", and read each digit separately after the decimal.

Examples:

$$3.14 \to \text{three point one four}, \quad 12.05 \to \text{twelve point zero five}$$

Practice: Write the spelling of the following decimal numbers.

- $4.5 \to$ ____________________
- $0.25 \to$ ____________________
- $10.75 \to$ ____________________
- $12.08 \to$ ____________________
- $100.01 \to$ ____________________

1.1.3 Reading Fractions

A fraction has the general form $\frac{a}{b}$, where the top number a is called the **numerator** and the bottom number b is the **denominator**.

A fraction is in simplest form when a and b have no common factors other than 1.

In reading fractions, the numerator is expressed as a cardinal number, while the denominator is expressed as an ordinal number followed by "s.

Example: $\frac{2}{8}$ is read *two-eighths*, and $\frac{3}{4}$ is *three fourths.*

Remarks:

- $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ are read *one half, one third, one quarter* respectively. These are exceptions to the general rule.
- For numerators greater than 1, make the denominator plural:
  $\frac{2}{3} \to$ two thirds, $\frac{5}{8} \to$ five eighths.
- If the numerator is less than the denominator the fraction is called a *proper fraction*; otherwise, it is called a *improper fraction.*
- 4. A mixed number is another way of expressing an improper fraction.
  $\frac{15}{4} = 3\frac{3}{4}$.

Practice:

- Write the spelling of the following fractions: $\frac{1}{8}, \frac{16}{5}, \frac{8}{10}, \frac{9}{4}$.
- In the fraction *two ninths*, __________ is the numerator, and __________ is the denominator.

1.1.4 Sets of Numbers

Numbers are classified into different sets according to their properties:

- **Natural Numbers (N):** Counting numbers $(1, 2, 3, \dots)$
- **Whole Numbers:** Natural numbers plus 0: $0, 1, 2, 3, \dots$
- **Integers (Z):** Whole numbers and their negatives $\dots, -2, -1, 0, 1, 2, \dots$
  -2 is read as "minus two" or "negative two."
- **Rational Numbers (Q):** numbers that can be written as a fraction $\frac{a}{b}$, where $b \neq 0$
- **Irrational Numbers:** Numbers that cannot be written as a fraction $\frac{a}{b}$ (e.g., π, $\sqrt{2}$, e)
- **Real Numbers (R):** all rational and irrational numbers
- **Complex Numbers (C):** numbers of the form $a + bi$, where $i = \sqrt{-1}$
  a is called the real part, b is called the imaginary part, and i is called imaginary unit of the complex number.

1.1.5 Factors and Multiples

Factor is a number that divides another number exactly (no remainder).
Example:
factors of $12 \to 1, 2, 3, 4, 6, 12$

Every number is divisible by 1 and itself; these are called **improper factors**. If there exist other factors, they are called **proper factors**.

Multiple is the result of multiplying a number by an integer.
Example:
multiples of 5ß$5, 10, 15, 20, 25, \ldots$

Divisibility: A number a is divisible by another number b if the division has no remainder.

Example:
4 divides 12, so 12 is divisible by 4. 4 is a factor of 12.

We call a number that is divisible by 2 an **even number**.

We call a number that is not divisible by 2 an **odd number**.

A **prime number** is a number that has only the two improper factors 1 and itself.

A **composite number** is a number that has more than the two improper factors 1 and itself.

1 is neither prime nor composite.

Perfect numbers are numbers that are equal to the sum of their proper factors
Examples:
$6 = 1 + 2 + 3$,
$28 = 1 + 2 + 4 + 7 + 14$)

Common Divisors are numbers that divide two or more given numbers exactly (with no remainder). The greatest of them is called the greatest common divisor (GCD) or highest common factor (HCF).

Examples:
$1, 2, 3, 4, 6$, and 12 are divisors (factors) of 12.
$1, 3, 5$, and 15 are divisors of 15. 1 and 3 are common divisors of 12 and 15.
3 is the greatest common divisor of 12 and 15. The g.c.d of 12 and 15 is 3. ($gcd(12, 15) = 3$.)

Common multiples are numbers that are multiples of two or more given numbers. The smallest of them is called the least common multiple (LCM).

Examples:
$5, 10, 15, 20, 25, \ldots$ are multiples of 5.
$4, 8, 12, 16, 20, 24, \ldots$ are multiples of 4.
$5, 10, 15, 20$ are four first multiples of 5.
$4, 8, 12, 16, 20$ are five first multiples of 4.
$20, 40, 60, \ldots$ are common multiples of 4 and 5.
20 is the least common multiple of 4 and 5.
The l.c.m of 4 and 5 is 20 (lcm(4,5) = 20.)

Problems

1. Write down the first five multiples of 12.

2. List all positive divisors of 84.

3. Find all common divisors of 48 and 180.

4. Reduce the fraction $\dfrac{156}{390}$ to its simplest form.

5. The sum of two numbers is 20 and their product is 96. Find the sum of their reciprocals.

6. Compute the product $\gcd(2574, 1302) \times \operatorname{lcm}(2574, 1302)$.

7. Let x be the smallest of three positive integers whose product is 360. What is the largest possible value of x?

8. If P is the product of all prime numbers less than 50, what is the units digit of P?

9. Find a positive integer that equals seven times the sum of its digits. (Find all such integers if more than one exists.)

10. What is $\gcd(924, 385)$?

11. The product of 791 and 807 is divided by the sum $29 + 71$. What is the remainder?

12. The average of four consecutive odd integers is 27. Find the largest of the four integers.

13. When the five-digit number $12N45$ is divided by 9, the remainder is 7. Find all possible digits N.

14. Find the least positive integer divisible by $8, 9$, and 15.

15. If n^2 has exactly 9 positive divisors, list all possible forms of n (up to prime powers structure).

16. Find all two-digit integers N such that N is divisible by the sum of its digits.

17. Show that among any 10 consecutive integers, there is exactly one divisible by 10. Explain.

18. A number M when divided by 7 leaves remainder 4, and when divided by 9 leaves remainder 2. Find the smallest positive M.

19. Let a, b, c be positive integers with $\gcd(a, b, c) = 1$ and $abc = 2^4 \cdot 3^2 \cdot 5$. List one possible triple (a, b, c) and justify your choice.

20. Find the remainder when 3^{100} is divided by 13.

1.2 Sequence and series

- A sequence is an ordered list

  $$\{a_i, a_{i+1}, a_{i+2}, \dots, a_n\}$$

  of numbers that follows a specific rule or pattern, where $i, n \in \mathbb{Z}$, and $n > i$.

  Each element in the list is called a term of the sequence.

  The general term of a sequence is denoted by a_n, where n indicates the position of the term.

  The term a_i is called the first term of the sequence.

  a_j, a_{j+1}, for any integer j are called consecutive terms.

  Example:

  $$\{1, \frac{1}{2}, \frac{1}{5}, \frac{1}{10} \dots .\}$$

  The general term of the sequence is given by

  $$a_n = \frac{1}{1+n^2}, n = 1, 2, \dots .$$

  $a_1 = 1$ is the first term.

  $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{5}$ are consecutive terms.

- Some Sequences are defined by a recursive relation, without an explicit formula for their general term.
  Example 1: *Babylonian method* (or Newron's method) for approximating square roots

  Defined by the recurrence relation: $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right), \quad a_1 = 1, n = 1, 2, 3, \dots$

  This sequence converges to $\sqrt{2}$.
  Example 2:

  $$a_n = \left(1 + \frac{1}{n}\right)^n$$

  As $n \to \infty$, we have

  $$a_n \to e.$$

  This sequence defines the **Euler number** e.

- Arithmetic sequence is a sequence with constant difference:

  $$a_{i+1} - a_i = d,$$

  d is called common difference.
  Example
  The odd numbers form an arithmetic sequence with a common difference of $d = 2$.

- Geometric sequence is a sequence with constant ratio:

$$\frac{a_{i+1}}{a_i} = r$$

  Example
  The sequence of powers of 10 $a_n = 10^n$, where n is integer, is a geometric sequence with common ratio $r = 10$. It is useful in scientific notation.

- Harmonic sequence is a sequence whose reciprocals form an arithmetic sequence.
  Example:

$$1, \frac{1}{2}, \frac{1}{3}, \dots .$$

  The harmonic sequence appears naturally in physics since the wavelengths of the harmonics are inversely proportional to integers

$$\lambda_n = \frac{\lambda_1}{n}.$$

  That's why they are called harmonics — their pattern follows a harmonic sequence.

- Limit of a sequence The value L that a_n approaches as $n \to +\infty$

- Monotonic sequence is a sequence that is always increasing or decreasing.

  The sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, ... is monotone increasing but not strictly monotone i

$$\text{The sequence } (a_n) = (n^3) \text{ is strictly monotone increasing.}$$

$$\text{The sequence } (a_n) = \left(\frac{1}{n}\right) \text{ is strictly monotone decreasing.}$$

$$\text{The sequence } (a_n) = ((-1)^{n+1}) \text{ is not monotone.}$$

- Bounded sequence is a sequence whose terms stay within fixed limits (above and below)

- Unbounded sequence is a sequence whose terms grow beyond any fixed limit.

- a sequence is said to be convergent if its terms approach a finite limit L as n tends to infinity.

- Divergent sequence A sequence that does not approach a finite limit L as n tends to infinity.

- Oscillating sequence is a sequence that neither converges nor diverges, but alternates between values.
  Example:

$$a_n = (-1)^n$$
$$a_n = \frac{(-1)^n}{n}$$
$$a_n = 1 + (-1)^n$$

- the partial sum is the sum of its first n terms. It is denoted by S_n and defined as:

$$S_n = \sum_{i=1}^{n} = a_1 + a_2 + a_3 + \cdots + a_n$$

- A series is the sum of the terms of a sequence.

$$S = \sum_{i=1}^{+\infty} = a_1 + a_2 + a_3 + \ldots$$

Examples

1.3 Logic and Sets

- A statement is a sentence that is either true or false. they are often represented by a lower case letters such as p,q or r.
Example:
Every odd number can be expressed as $2k + 1$, where k is an integer ($k \in \mathbb{Z}$).

- An argument is a group of statements in which some statements (called premises) are given to support another statement (called the conclusion).

- Conditional (implication): An 'if ... then ... ' sentence, i.e., a sentence that expresses some kind of conditional relationship between the two parts of the sentence.
The first part (the if-clause) of a conditional statement is called the antecedent, and the second part (the then-clause) is called the consequent.
Example:

$$\forall x \in \mathbb{R},\ (x > 0 \Rightarrow -x < 0).$$

$$\forall n \in \mathbb{Z},\ (n \text{ is even} \Rightarrow n^2 \text{ is even}).$$

- Compound statement is a statement made by joining two or more simple statements using words like and, or, or if...then.

- Disjunction statement is a compound statement that use the word or.

- Conjunction statement A compound statement that use the word and.

- A negation is a statement that has the opposite truth value of the original statement, usually formed by adding the word "not."
Reading common logic symbols

$\forall$	for all for every
$\exists$	there exists for some
$\neg p$	not p the negation of p
$p \wedge q$	p and q the conjunction of p and q
$p \vee p$	p or q the disjunction of p or q
$p \Rightarrow q$	if p, then q p implies q
$p \Leftrightarrow q$	p if and only if q p is equivalent to q

- Set is collection of objects which have same or equal characteristic. The objects in a set are called elements or members of the set. The set of all subsets of a given set A is called a power set.

- The number of distinct objects in a set is called the cardinality of the set.
  Example:
  The cardinality of empty set is zero.

- Complementary of a Set
  Let U be the **universal set** (the set of all elements under consideration), and let A be a **subset** of U.

  The **complementary set** of A, denoted by

  $$A^c \quad \text{or} \quad \overline{A},$$

  is the set of all elements in U that are **not in** A.

  $$A^c = \{x \in U \mid x \notin A\}$$

Example

If

$$U = \{1, 2, 3, 4, 5\}, \quad A = \{1, 3, 5\},$$

then

$$A^c = \{2, 4\}.$$

- A set is called finite if its cardinality (number of elements) is finite; otherwise, it is called infinite.
  Examples:
  1. The set of solutions to the equation

  $$x + y = 4, \quad x, y \in \mathbb{N},$$

is given by

$$S = \{(0,4),(1,3),(2,2),(3,1),(4,0)\},$$

which is a finite set consisting of 5 ordered pairs.

2. The set of complex roots of unit

$$U_n = \left\{ e^{\frac{2\pi i k}{n}} \mid k = 0,1,2,\ldots,n-1 \right\}$$

For example,

$$U_4 = \{1,\, i,\, -1,\, -i\}.$$

This is a finite set on the complex plane, important in Fourier analysis.

- A set is bounded if all its elements lie within some fixed limits — that is, it can be enclosed between two real numbers. Otherwise, it is called unbounded.
  Example:
  1. Set of trigonometric values:

  $$A = \{\sin(x), x \in \mathbb{R}\}$$

  is bounded by $[-1,1]$.
  2. Exponential growth set:

  $$A = \{e^n : n \in \mathbb{N}\}$$

  is unbounded set.

- A set is called **open** if it does not contain its boundary points, and **closed** if it contains all of its boundary (limit) points.
  Example:
  1. Open ball in $C([0,1])$

  $$B(f_0,\epsilon) = \{f \in C([0,1]) : \|f - f_0\|_\infty < \epsilon\}$$

  open in the sup norm. Every function within ϵ of f_0 is included.
  2.

  $$F = \{f \in C([0,1]) : \|f\|_\infty \leq 1\}$$

  Closed because limits of functions in F remain bounded by 1.

- Some sets can be both open and closed (called clopen), such as the empty set $\varnothing$, and the whole real line $\mathbb{R}$.

- A countable set is a set whose elements can be listed one by one. If this is not possible, the set is called uncountable.
  Example:
  1. The set of prime numbers is a countable set.
  2. The power set of natural numbers is uncountable set.

Reading common sets symbols

$x \in A$	x belongs to A
$x \notin A$	x does not belong to A
$A \subset B$	A is subset of B A is contained in B
$A \subseteq B$	A is subset of or equal to B
$A \supset B$	A is superset of B
$A \cup B$	A union B
$A \cap B$	A intersection B
$A \setminus B$	A minus B the difference of A and B
$A \times B$	A cross B the cartesian product of A and B

Logic Exercise

Fill in the blanks with the correct words or answer the questions.

1. A ______________ is a sentence that is either __________ or false.
2. If a statement is true, then its ______________ is false.
3. A compound statement that uses the word "or" is called a ______________.
4. A compound statement that is true only when all the statements that compose it are true is called a ______________.
5. If the antecedent is true and the ______________ is false, then an ______________ is false.
6. What is the name of a compound statement that is true if the two statements being combined have the same truth value?
7. Given two statements P and Q, where P is true — what can you say about the truth value of their *disjunction* and *implication*?
8. An ______________ contains __________ and a conclusion.

Exercise: Classify the Following Sets

For each of the following sets, determine whether it is:
(a) Bounded or Unbounded
(b) Open or Closed
(c) Finite or Infinite.
(d) Countable or Uncountable.

Give a short justification for each answer.

1. $A = [0, 5]$
2. $B = (0, 5)$

3. $C = \mathbb{N} = \{1, 2, 3, \dots\}$
4. $D = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$
5. $E = \mathbb{R}$
6. $F = \mathbb{Q} \cap [0, 1]$
7. $G = \{e^n : n \in \mathbb{N}\}$
8. $H = \mathcal{P}(\mathbb{N})$
9. $I = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$
10. $J = \{\sqrt{2}, \pi, e\}$

Exercise: Sets

Fill in the blanks with the correct words or answer the questions.

1. if x belongs to A and B, so x belongs to ______________
2. x belongs to the union of A and B, it means that______________________________
3. x belongs to $\mathbb{R}$ and does not belong to $\mathbb{Q}$, so it belongs to ______________________________ ,and it is called ______________
4. Let A the set of the odd number less than 30, B the set of prime number greater than 8, so the cardinality of the intersection of A and B is ______________

Chapter 2

Reading Mathematics expressions

practice writing mathematically correct sentences

2.1 Arithmetic Operations

Arithmetic operations are the basic rules we use to work with numbers. There are four main operations:
 Addition
 Subtraction
 Multiplication
 Division

2.1.1 Addition

Addition is the operation of combining two or more numbers to get a total (sum).

Notation	a	$+$	b	$=$	c
Terms	addend	addition sign	addend	equality sign	sum
Reading	a	plus	b	equals	c
	a	and	b	is	c

Property	**Expression**
Commutative	$a + b = b + a$
Associative	$(a + b) + c = a + (b + c)$
Identity Element	$a + 0 = 0 + a = a$

2.1.2 Subtraction

Subtraction is the operation of finding the difference between two numbers.

Notation	a	$-$	b	$=$	c
Terms	minuend	subtraction sign	subtrahend	equality sign	difference
Reading	a	minus	b	equals	c
	a	take away	b	is	c

Properties

Subtraction is neither associative nor commutative, and it does not have an identity element.

2.1.3 Multiplication

Multiplication is the operation of combining equal groups of a number to get a total (Repeated addition).

Notation	a	$\times$	b	$=$	c
Terms	multiplier	multiplication sign	multiplicand	equality sign	product
Reading	a	times	b	equals	c
	a	multiplied by	b	is	c

Property	**Expression**
Commutative	$a \times b = b \times a$
Associative	$(a \times b) \times c = a \times (b \times c)$
Distributive	$a \times (b + c) = a \times b + a \times c$
Identity	$a \times 1 = 1 \times a = a$
Zero Property	$a \times 0 = 0 \times a = 0$

2.1.4 Division

Division is the operation of splitting a number into equal parts or finding how many times one number is contained in another to get a quotient. .

Notation	a	$\div$	b	$=$	c
Terms	dividend	division sign	divisor	equality sign	quotient
Reading	a	divided by	b	equals	c
	a	over	b	is	c

Properties:

Division is neither associative nor commutative, and it does not have an identity element.

Zero Property

Division by zero is undefined:

$$\frac{a}{0} \text{ is not defined}$$

However,

$$\frac{0}{a} = 0, \text{ for any } a \neq 0$$

2.2 Comparing numbers

$=$	equals / is equals to
$\neq$	does not equal / is not equal to
$\approx$	is approximately equal to
$>$	is greater than
$<$	is less than
$\geq$	is greater than or equal
$\leq$	is less than or equal

Note There are other similar symbols to $\approx$, each with a small difference in meaning.

- $\cong \rightarrow$ for **congruence** or **almost equal** (geometry or arithmetic).
  Example: $\triangle ABC \cong \triangle DEF$ (The two triangles are congruent — same shape and size.)

- $\sim \rightarrow$ for **similarity** (same shape, ratio, or behavior).

  Example: $\triangle ABC \sim \triangle PQR$ (The two triangles are similar — same shape but different sizes.)

- $\simeq \rightarrow$ for **approximation** in **limits or functions**.
  Example: $\sin x \simeq x$ when $x \rightarrow 0$ (The sine of x is approximately equal to x for small values.)

The mathematical sentences that use symbols $=$ are called equation, and the mathematical sentences that use symbols $<, >, \leq$ or $, \geq$ are called inequalities.
Examples
$ax + b = 0$ is a linear equation.
$\dfrac{a+b}{2} \geq \sqrt{ab}$ is called AM-GM inequality
Exercise 1:
Read the following equations

a) $1.309 + 115 = 116.3090$

b) $40056 - 520 = 39536$

c) $-111 \times 25 = -2775$

d) $\dfrac{2.15}{3} = 0.7167 \approx 0.72$

Exercise 2:

Complete and write and the math expressions.

1. The ______________ of one third and two eights is seven twelfths.

2. The total of eight, two, and five is ______________
3. If you subtract two and a half from six, you get ______________
4. The difference between five sixths and one third is ______________
5. 20 minus 5.5 equals ______________
6. Six times 1.2 equals ______________
7. The product of 2.5 and four is ______________
8. The ______________ of 2.5 and 4 gives 10.
9. The quotient of three fourths and one half is ______________
10. Twelve divided by three equals ______________

Exercise 3:
Fill the blank spaces with the right words

1. The product of two consecutive numbers is ______________ the sum of them, if ______________ are greater than ______________.
2. if the divisor is ______________ the dividend the quotient will be ______________ 1.
3. The value of π is ______________ $\frac{22}{7}$, that means that the ______________ is very small.
4. The area of a circle π the radius squared

Exercise 4:

2.3 Reading common functions

2.3.1 Powers

A power (or exponent) represents repeated multiplication of a number by itself, and it is written as a^n.
In this expression, a is the base and n is the exponent.
It is read as: a to the nth power (or a to the n, or a raised to n).
Example:

$$2^4 = 2 \times 2 \times 2 \times 2 = 16$$

In this example "2" is the base and "4" is the exponent.
Exmaples

Expression	Reading
x^2	x squared or, x to the 2nd power
x^3	x cubed
x^4	x to the fourth power

Reading numbers with powers

Expression	Reading
2^2	two squared
4^3	four cubed
10^4	ten to the fourth power
y^{10}	y to the tenth power

Reading special cases

Expression	Reading
x^{-2}	x to the negative two
$x^{1/2}$	x to the one-half
$x^{1/3}$	x to the one-third
2^{x+1}	two to the power of x plus one
$16^{3/2}$	sixteen to the three-halves
$(x+y)^{3/5}$	x plus y, all raised to the three-fifths power or, x plus y, all to the three fifths

2.3.2 Rules of Powers

The **rules of powers** are essential tools that allow us to simplify and manipulate expressions efficiently. They are widely used in algebra, calculus, physics, and many areas of science and engineering.

Product of Powers Rule

The product of two powers with equal bases equals the base raised to the sum of the two exponents.

$$a^m \cdot a^n = a^{m+n}$$

Example:

$$2^3 \cdot 2^4 = 2^{3+4} = 2^7 = 128$$

Reading: a to the m times a to the n equals a to the m plus n

Quotient of Powers Rule

The quotient of two powers with equal bases equals the base raised to the difference of the exponents.

$$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$$

Example:

$$\frac{5^6}{5^2} = 5^{6-2} = 5^4 = 625$$

Reading: a to the m divided by a to the n equals a to the m minus n.

Power of a Power Rule

Raising a power to another exponent equals the base raised to the product of the two exponents.

$$(a^m)^n = a^{m \cdot n}$$

Example:

$$(3^2)^4 = 3^{2 \cdot 4} = 3^8 = 6561$$

Reading: a to the m, raised to the n, equals a to the m times n.

Power of a Product Rule A product raised to a power is equal to the product of each base raised to the same exponent.

$$(ab)^n = a^n \cdot b^n$$

Example:

$$(2 \cdot 3)^4 = 2^4 \cdot 3^4 = 16 \cdot 81 = 1296$$

Reading: a times b raised to the n equals a to the n times b to the n.

Power of a Quotient Rule A quotient raised to a power is equal to the quotient of each factor raised to that power.

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, \quad b \neq 0$$

Example:

$$\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2} = \frac{4}{9}$$

Reading: a over b raised to the n equals a to the n over b to the n.

Zero Exponent Rule Any non-zero number raised to the zero power equals one.

$$a^0 = 1, \quad a \neq 0$$

Example:

$$7^0 = 1$$

Reading: a to the zero power equals one.

Negative Exponent Rule A negative exponent indicates the reciprocal of the base raised to the corresponding positive exponent.

$$a^{-n} = \frac{1}{a^n}, \quad a \neq 0$$

Example:

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

Reading: a to the power of negative n equals one over a to the n.

Fractional Exponent Rule A fractional exponent represents both a root and a power. The denominator indicates the root, and the numerator indicates the power.

$$a^{m/n} = (\sqrt[n]{a})^m$$

Example:

$$8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$$

Reading: a to the m over n or the n-th root of a, raised to the m.

2.3.3 Roots

A **root** of a number is a value that, when raised to a certain power, gives the original number. The most common roots are the **square root** and the **cube root**, but in general we can define the **nth root** of a number.

Notation: The square root of a is written as $\sqrt{a}$, and the nth root of a is written as $\sqrt[n]{a}$.

$$\sqrt[n]{a} = b \quad \text{means} \quad b^n = a$$

Reading roots:

Expression	Reading
$\sqrt{4}$	square root of four
$\sqrt{25}$	square root of twenty-five
$\sqrt[3]{8}$	cube root of eight
$\sqrt[5]{32}$	fifth root of thirty-two
$\sqrt[n]{x}$	nth root of x

Examples:

$$\sqrt{9} = 3, \quad \sqrt[3]{27} = 3, \quad \sqrt[4]{16} = 2$$

Rules of Roots

Product Rule

The root of a product is equal to the product of the roots:

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}, \quad a \geq 0, b \geq 0$$

Example:

$$\sqrt{16 \cdot 9} = \sqrt{16} \cdot \sqrt{9} = 4 \cdot 3 = 12$$

Reading: "The root of a times b equals the root of a times the root of b."

Quotient Rule

The root of a quotient is equal to the quotient of the roots:

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}, \quad a \geq 0, b > 0$$

Example:

$$\sqrt{\frac{25}{9}} = \frac{\sqrt{25}}{\sqrt{9}} = \frac{5}{3}$$

Reading: "The root of a over b equals the root of a over the root of b."

Power and Root Rule

Raising a root to its corresponding power returns the original number:

$$(\sqrt[n]{a})^n = a, \quad a \geq 0$$

Example:

$$(\sqrt[3]{8})^3 = 8$$

Reading: "The nth root of a, raised to the nth power, equals a."

Root of a Power

The nth root of a number raised to a power equals the number raised to the fraction of that power over n:

$$\sqrt[n]{a^m} = a^{m/n}, \quad a \geq 0$$

Example:

$$\sqrt[3]{8^2} = 8^{2/3} = 4$$

Reading: "The nth root of a to the m equals a to the m over n."

Square Root of a Square

The square root of a squared number returns the absolute value of the original number:

$$\sqrt{x^2} = |x|$$

Example:

$$\sqrt{(-5)^2} = \sqrt{25} = 5$$

Reading: "The square root of x squared equals the absolute value of x."

Exercises: Powers and Roots

1. Two to the ______________ power equals eight
2. Ten to the third times ten to the negative fifth equals ten to the ______________
3. the reciprocal of ten ______________ equals one hundred
4. y squared, ______________ equals y to the eighth.
5. The ______________ root of twenty-seven equals three
6. The square root of ______________ equals one-half
7. The square root of sixteen ______________ five equals nine.
8. Two to the third power, ______________ equals sixty-four.
9. Five squared ______________ three squared equals sixteen
10. The cube root of eight ______________ two equals four.
11. The square root of one plus ______________ equals square root of three
12. Two cubed plus four divided by the square root of ______________ equals six.
13. x to the one-half times y to the one-third, ______________ equals x cubed y squared.

2.3.4 Logarithms

A **logarithm** is the inverse operation of exponentiation. In other words, if $b^y = x$, then the logarithm base b of x is y:

$$\log_b(x) = y \quad \text{means} \quad b^y = x, \quad b > 0, b \neq 1, x > 0$$

The most common logarithm used in mathematics is the natural logarithm, which has base $e \approx 2.71828$. It is written as $\ln(x)$, and it satisfies the relation $e^{\ln(x)} = x$.

Reading logarithms:

Expression	Reading
$\log_2 8$	logarithm base 2 of 8
$\log_{10} 100$	logarithm base 10 of 100
$\ln 7$	natural logarithm of 7
$\log_b x$	logarithm base b of x

Examples:

$$\log_2 8 = 3 \quad \text{because} \quad 2^3 = 8$$

$$\log_{10} 1000 = 3 \quad \text{because} \quad 10^3 = 1000$$

$$\ln e^2 = 2 \quad \text{because} \quad e^2 = e^2$$

Rules of Logarithms

Product Rule

The logarithm of a product equals the sum of the logarithms:

$$\log_b(xy) = \log_b x + \log_b y$$

Example:

$$\log_2(8 \cdot 4) = \log_2 8 + \log_2 4 = 3 + 2 = 5$$

Reading: "The logarithm of x times y equals the logarithm of x plus the logarithm of y."

Quotient Rule

The logarithm of a quotient equals the difference of the logarithms:

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

Example:

$$\log_2 \frac{16}{4} = \log_2 16 - \log_2 4 = 4 - 2 = 2$$

Reading: "The logarithm of x over y equals the logarithm of x minus the logarithm of y."

Power Rule

The logarithm of a power equals the exponent multiplied by the logarithm of the base:

$$\log_b(x^n) = n \cdot \log_b x$$

Example:

$$\log_2 8^3 = 3 \cdot \log_2 8 = 3 \cdot 3 = 9$$

Reading: "The logarithm of x to the n equals n times the logarithm of x."

Change of Base Rule

A logarithm can be expressed in terms of another base:

$$\log_b x = \frac{\log_k x}{\log_k b}, \quad b > 0, b \neq 1, k > 0, k \neq 1$$

Example:

$$\log_2 8 = \frac{\log_{10} 8}{\log_{10} 2} = \frac{0.9031}{0.3010} = 3$$

Reading: "The logarithm of x base b equals the logarithm of x base k divided by the logarithm of b base k."

Special Cases

$$\log_b 1 = 0 \quad \text{because} \quad b^0 = 1$$
$$\log_b b = 1 \quad \text{because} \quad b^1 = b$$

Reading: The logarithm of 1 base b equals zero and The logarithm of b base b equals one.

2.3.5 Exponential Function

An **exponential function** is a mathematical function in which a constant base a is raised to a variable exponent x. That is, it has the form

$$f(x) = a^x,$$

where $a > 0$ and $a \neq 1$. The most commonly used base in mathematics is the number $e \approx 2.71828$, which leads to the **natural exponential function**, written as

$$f(x) = e^x \quad \text{or} \quad f(x) = \exp(x).$$

This function has the important property that its derivative is equal to itself:

$$\frac{d}{dx}e^x = e^x.$$

Reading exponentials:

Expression	Reading
e^x	the exponential of x e to the x
e^2	e squared
e^{-1}	the exponential of negative one e to the of the negative one
$\exp(3)$	e to the three
e^{x+1}	e to the x plus one
$\exp(a - b)$	e to the a minus b

Examples:

$$e^0 = 1, \quad e^1 = e \approx 2.718, \quad e^2 \approx 7.389$$

Rules of Exponential Functions

Product Rule
Multiplying exponentials with the same base adds the exponents:

$$e^x \cdot e^y = e^{x+y}$$

Example:

$$e^2 \cdot e^3 = e^{2+3} = e^5$$

Reading: "e to the x times e to the y equals e to the x plus y."

Quotient Rule
Dividing exponentials with the same base subtracts the exponents:

$$\frac{e^x}{e^y} = e^{x-y}$$

Example:

$$\frac{e^5}{e^2} = e^{5-2} = e^3$$

Reading: "e to the x divided by e to the y equals e to the x minus y."

Power of an Exponential
Raising an exponential to another power multiplies the exponents:

$$(e^x)^n = e^{x \cdot n}$$

Example:

$$(e^2)^3 = e^{2 \cdot 3} = e^6$$

Reading: "e to the x, raised to the n, equals e to the x times n."

Inverse Relation with Logarithms
The exponential function and the natural logarithm are inverses:

$$\ln(e^x) = x, \quad e^{\ln x} = x, \quad x > 0$$

Example:

$$\ln(e^3) = 3, \quad e^{\ln 5} = 5$$

Reading: "The natural logarithm of e to the x equals x" and "e to the natural logarithm of x equals x."

Special Cases

$$e^0 = 1$$
$$e^1 = e$$

Reading: "e to the zero equals one" and "e to the first power equals e."

Exercises: Logarithm and Exponential

Fill in the blanks with the correct words, then rewrite each sentence as a mathematical expression.

1. The logarithm base ten of _______ equals two.

2. The natural logarithm of _______ equals zero.

3. The logarithm base three of twenty-seven equals _______.

4. e raised to the zero power equals _______.

5. The natural logarithm of one over e equals _______.

6. The logarithm base two of one-half equals _______.

7. The expression "log of a product" means log of a times b equals $\log(a)$ _______

8. The exponential of the natural log of x equals _______.

9. The logarithm base five of one equals _______.

10. The natural logarithm of the square root of x equals one-half _______.

11. The exponential of negative two equals one over e _______.

12. The logarithm base ten of ten-thousand equals _______.

13. The natural logarithm of e cubed equals _______.

14. The equation ten raised to the log base ten of x equals _______.

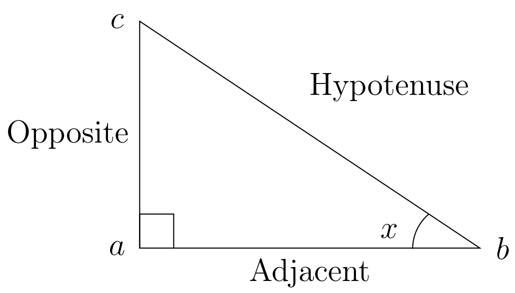
2.3.6 Trigonometric Functions

Definition: Trigonometric functions relate the angles of a right triangle to the ratios of its sides.

Right Triangle Terminology

- **Hypotenuse** : always the longest side of a right triangle, opposite the right angle.
- **Opposite side** : the side directly across from the angle you are considering (e.g., angle b).
- **Adjacent side** : the side next to the angle you are considering (angle b), but not the hypotenuse.

Diagram



The most common trigonometric functions are:

$$\sin x, \quad \cos x, \quad \tan x, \quad \cot x, \quad \sec x, \quad \csc x$$

Practice:
Using the right triangle diagram below, write the expression of each trigonometric function $\sin a$, $\cos a$, $\tan a$, $\cot a$, $\sec a$ and $\csc a$ in terms of the triangle's sides: **opposite**, **adjacent**, and **hypotenuse**.

Reading trigonometric functions:

Expression	Reading
$\sin x$	sine of x
$\cos x$	cosine of x
$\tan x$	tangent of x
$\cot x$	cotangent of x
$\sec x$	secant of x
$\csc x$	cosecant of x

Examples of function values:

$$\sin 0 = 0, \quad \cos 0 = 1, \quad \tan 0 = 0$$

$$\sin \frac{\pi}{2} = 1, \quad \cos \frac{\pi}{2} = 0, \quad \tan \frac{\pi}{4} = 1$$

Basic Trigonometric Identities

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Reading: "Sine squared x plus cosine squared x equals one," "one plus tangent squared x equals secant squared x," "one plus cotangent squared x equals cosecant squared x."

Reciprocal Identities

$$\csc x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \cot x = \frac{1}{\tan x}$$

Reading: "Cosecant x equals one over sine x," "secant x equals one over cosine x," "cotangent x equals one over tangent x."

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}$$

Reading: "Tangent x equals sine x over cosine x," "cotangent x equals cosine x over sine x."

Co-Function Identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x, \quad \cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x, \quad \cot\left(\frac{\pi}{2} - x\right) = \tan x$$

Reading: "Sine of pi over two minus x equals cosine x," etc.

Reading: "Sine is an odd function," "cosine is an even function," "tangent is an odd function."

Angle Addition and Subtraction Identities

These identities involve the sum or difference of two angles, a and b.

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

Reflection Identities

These identities involve flipping the angle over an axis:

- $-x \rightarrow$ reflection across the x-axis (it moves the angle from the first quadrant to the fourth quadrant) **Examples:**

$$\sin(-x) = -\sin x, \quad \cos(-x) = \cos x, \quad \tan(-x) = -\tan x$$

- $\pi - x \rightarrow$ reflection across the vertical line $x = \dfrac{\pi}{2}$ (it moves the angle from the first quadrant to the second quadrant)
  Examples

$$\sin(\pi - x) = \sin x, \quad \cos(\pi - x) = -\cos x$$

$$\tan(\pi - x) = -\tan x, \quad \cot(\pi - x) = -\cot x$$

- $\pi + x \rightarrow$ reflection through the origin (it moves the angle from the first quadrant to the third quadrant)
  Examples

$$\sin(\pi + x) = -\sin x, \quad \cos(\pi + x) = -\cos x$$

$$\tan(\pi + x) = \tan x, \quad \cot(\pi + x) = \cot x$$

Special Angles Table

x	$\sin x$	$\cos x$	$\tan x$
0	0	1	0
$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	1	0	undefined

Exercise 2. ***Evaluate Using Reflection Identities***

Evaluate the following expressions ***without a calculator****:*

1. $\sin\left(\pi - \frac{\pi}{6}\right)$
2. $\cos\left(-\frac{\pi}{4}\right)$
3. $\tan\left(\pi + \frac{\pi}{3}\right)$
4. $\sin\left(-\frac{2\pi}{3}\right)$
5. $\cos\left(\pi + \frac{\pi}{6}\right)$

Exercise 3. 1. *Solve the equation for* $0^\circ < x < 360^\circ$*:*

$$\cos x = -\frac{1}{2}.$$

2. *Solve the equation for* $0^\circ < x < 720^\circ$*:*

$$\sin(2x) = \frac{3}{2}.$$

3. *Find all angles* x *in* $0^\circ < x < 540^\circ$ *such that:*

$$\tan x = 1.$$

2.3.7 Inverse Trigonometric Functions

Inverse trigonometric functions return the angle whose trigonometric function has a given value. They are the inverses of the standard trigonometric functions on their principal domains.

Definition:

$$y = \sin x \implies x = \arcsin y, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$
$$y = \cos x \implies x = \arccos y, \quad x \in [0, \pi]$$
$$y = \tan x \implies x = \arctan y, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
$$y = \cot x \implies x = \operatorname{arccot} y, \quad x \in (0, \pi)$$
$$y = \sec x \implies x = \operatorname{arcsec} y, \quad x \in [0, \pi], x \neq \pi/2$$
$$y = \csc x \implies x = \operatorname{arccsc} y, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], x \neq 0$$

Reading inverse trigonometric functions:

Expression	Reading
$\arcsin x$	arcsine of x
$\arccos x$	arccosine of x
$\arctan x$	arctangent of x
$\operatorname{arccot} x$	arccotangent of x
$\operatorname{arcsec} x$	arcsecant of x
$\operatorname{arccsc} x$	arccosecant of x

Examples:

$$\arcsin\frac{1}{2} = \frac{\pi}{6}, \quad \arccos 0 = \frac{\pi}{2}, \quad \arctan 1 = \frac{\pi}{4}$$

Domain and Range Table:

Function	Domain	Range
$\arcsin x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\arccos x$	$[-1, 1]$	$[0, \pi]$
$\arctan x$	$\mathbb{R}$	$(-\pi/2, \pi/2)$
$\operatorname{arccot} x$	$\mathbb{R}$	$(0, \pi)$
$\operatorname{arcsec} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi], x \neq \pi/2$
$\operatorname{arccsc} x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, \pi/2], x \neq 0$

Reading Examples:

Expression	Reading
$\arcsin 1 = \pi/2$	"arcsine of one equals pi over two"
$\arccos 0 = \pi/2$	"arccosine of zero equals pi over two"
$\arctan 1 = \pi/4$	"arctangent of one equals pi over four"

Exercise 4. ***Inverse Trigonometric Functions Exercises***

Part A — Basic Evaluation

Evaluate (exact values):

1. $\arcsin(1)$

2. $\arcsin\left(\frac{\sqrt{3}}{2}\right)$

3. $\arccos(0)$

4. $\arccos\left(-\frac{1}{2}\right)$

5. $\arctan(1)$

6. $\arctan(-\sqrt{3})$

Part B — Solve for x

Solve for x in the principal range:

1. $\arcsin x = \frac{\pi}{6}$

2. $\arccos x = \frac{2\pi}{3}$

3. $\arctan x = 1$

Solve for x in $0 \le x \le 2\pi$:

1. $\sin x = \frac{1}{2}$
2. $\cos x = -\frac{\sqrt{2}}{2}$
3. $\tan x = \sqrt{3}$

Part C — Simplify Expressions Using Inverse Trig Identities
Simplify:

1. $\arcsin(\sin \frac{5\pi}{6})$
2. $\arccos(\cos \frac{7\pi}{4})$
3. $\arctan(\tan(-\frac{\pi}{3}))$

Express in simplest form:

1. $\sin(\arccos \frac{3}{5})$
2. $\cos(\arcsin \frac{4}{5})$
3. $\tan(\arcsin \frac{2}{3})$

Part D — Equations Involving Inverse Trig Functions
Solve for x:

1. $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$
2. $\arctan(x) + \arctan(2x) = \frac{\pi}{4}$
3. $\arcsin(2x) = \frac{\pi}{6}$

Part E — Word Problems / Application

1. *If $\theta = \arccos \frac{12}{13}$, find $\sin\theta$ and $\tan\theta$.*
2. *A right triangle has opposite side = 5 and hypotenuse = 13. Find the angle using an inverse trig function.*
3. *Prove that for any x in the principal range:*

$$\arcsin x + \arccos x = \frac{\pi}{2}.$$

4. *If $\tan(\arcsin x) = \frac{3}{4}$, find x.*

2.4 Reading Common Operators

Mathematical operators such as **limits**, **derivatives**, and **integrals** are essential in calculus. This handout explains how to *read* these expressions clearly and correctly and gives rules, examples and exercises.

2.4.1 Limits

A **limit** describes the value that a function approaches as the input approaches a specific value.

$$\lim_{x \to a} f(x)$$

Reading limits:

Expression	Reading
$\lim_{x \to 2} f(x)$	the limit of $f(x)$ as x approaches 2 the limit of $f(x)$ as x tends to 2
$\lim_{x \to 0} \frac{\sin x}{x}$	the limit of sine x over x as x approaches zero
$\lim_{t \to \infty} g(t)$	the limit of $g(t)$ as t approaches infinity
$\lim_{x \to a^-} f(x)$	the limit of $f(x)$ as x approaches a from the left / below The left-hand limit of f x
$\lim_{x \to a^+} f(x)$	the limit of $f(x)$ as x approaches a from the right / above The right-hand limit of f x

Examples:

$$\lim_{x \to 1}(3x + 2) = 5$$

Reading: the limit of three x plus two as x approaches one equals five.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Reading: the limit of sine x over x as x approaches zero equals one.

2.4.2 Method for Computing Limits

- Direct Substitution Rule
  If a function is continuous at a given point, then the limit at that point can be found simply by substituting the value into the function.

- **General Limit Rules**

  1. **Sum Rule**
     The limit of the sum of two functions equals the sum of their individual limits.

$$\lim_{x \to c}[f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$

2. **Difference Rule**
   The limit of the difference of two functions equals the difference of their individual limits.
   $$\lim_{x\to c}[f(x) - g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$$
3. **Product Rule**
   The limit of the product of two functions equals the product of their individual limits.
   $$\lim_{x\to c}[f(x) \cdot g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$$
4. **Constant Multiple Rule**
   The limit of a constant multiplied by a function equals the constant multiplied by the limit of the function.
   $$\lim_{x\to c}[k \cdot f(x)] = k \cdot \lim_{x\to c} f(x), \quad k \text{ constant}$$
5. **Quotient Rule**
   The limit of the quotient of two functions equals the quotient of their limits, provided the limit of the denominator is not zero.
   $$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}, \quad \text{if } \lim_{x\to c} g(x) \neq 0$$

- **Factoring Method**
  The factoring method is a technique used to compute limits when direct substitution gives an indeterminate form such as $\frac{0}{0}$ The idea is to factor the numerator and/or denominator to cancel a common factor that causes the indeterminacy.

  Example Compute
  $$\lim_{x\to 3} \frac{x^2 - 9}{x - 3}.$$
  First, using the direct substitution gives
  $$\frac{3^2 - 9}{3 - 3} = \frac{0}{0} \quad \text{(indeterminate form)}$$
  Then we try with factoring the expression
  $$x^2 - 9 = (x - 3)(x + 3).$$
  So,
  $$\frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3}.$$
  Cancel the common factor
  $$\frac{(x - 3)(x + 3)}{x - 3} = x + 3.$$
  Finally, evaluate the limit
  $$\lim_{x\to 3}(x + 3) = 6.$$

- **Conjugate Method**

  The conjugate method is a technique used to compute limits, especially when the expression involves square roots and direct substitution results in an indeterminate form such as $\frac{0}{0}$. The idea is to multiply the numerator and denominator by the conjugate expression in order to simplify the radical and eliminate the indeterminacy.

  Example

  Compute
  $$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}.$$

  First, using direct substitution gives
  $$\frac{\sqrt{4}-2}{4-4} = \frac{0}{0} \quad \text{(indeterminate form)}.$$

  To remove the indeterminacy, multiply the numerator and denominator by the conjugate of the numerator:
  $$\frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2}.$$

  Simplify the numerator using the identity $(a-b)(a+b) = a^2 - b^2$:
  $$(\sqrt{x}-2)(\sqrt{x}+2) = x-4.$$

  So the expression becomes
  $$\frac{x-4}{(x-4)(\sqrt{x}+2)}.$$

  Cancel the common factor (valid for $x \neq 4$):
  $$\frac{x-4}{(x-4)(\sqrt{x}+2)} = \frac{1}{\sqrt{x}+2}.$$

  Finally, evaluate the limit:
  $$\lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}.$$

- **Squeeze Theorem**

  The Squeeze Theorem is used to compute limits when a function is difficult to evaluate directly but can be bounded between two simpler functions whose limits are known. If
  $$g(x) \le f(x) \le h(x)$$
  and
  $$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L,$$
  then
  $$\lim_{x\to a} f(x) = L.$$

  Example

Compute

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right).$$

We know that

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1.$$

Multiplying each part by x^2 (always positive),

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2.$$

Since

$$\lim_{x\to 0}(-x^2) = 0 \qquad \text{and} \qquad \lim_{x\to 0} x^2 = 0,$$

the Squeeze Theorem gives

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

- **L'Hôpital's Rule**

L'Hôpital's Rule applies to limits that result in an **indeterminate form** of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Statement: Let f and g be functions defined on an open interval I, and differentiable on $I \setminus \{c\}$, where c is a (possibly infinite) accumulation point of I. If

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0 \quad \text{or} \quad \pm\infty,$$

and

$$g'(x) \neq 0 \quad \text{for all } x \in I \setminus \{c\},$$

and if

$$\lim_{x\to c} \frac{f'(x)}{g'(x)}$$

exists (finite or infinite), then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

Differentiating the numerator and denominator often simplifies the quotient or converts it to a limit that can be directly evaluated using direct substitution.

Example 1: Indeterminate Form $\frac{0}{0}$

Compute

$$\lim_{x\to 0} \frac{\sin x}{x}.$$

First, using the direct substitution gives

$$\frac{\sin 0}{0} = \frac{0}{0} \quad \text{(indeterminate form)}$$

Using L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

Example 2: Indeterminate Form $\frac{\infty}{\infty}$

Compute

$$\lim_{x\to\infty} \frac{\ln x}{x}.$$

First, using the direct substitution gives

$$\frac{\ln(+\infty)}{+\infty} = \frac{+\infty}{+\infty}.$$

Using L'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

2.5 Further Topics

2.5.1 Other Indeterminate Forms Examples

Example 1: Indeterminate Form 1^∞

Compute the limit:

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x.$$

As $x \to \infty$, we have:

$$1 + \frac{1}{x} \to 1, \quad x \to \infty,$$

so the expression is of the indeterminate form 1^∞.

Let

$$y = \left(1 + \frac{1}{x}\right)^x.$$

Taking the natural logarithm,

$$\ln y = x \ln\left(1 + \frac{1}{x}\right).$$

Using the standard limit

$$\lim_{x\to\infty} x \ln\left(1 + \frac{1}{x}\right) = 1,$$

we obtain

$$\ln y = 1 \quad \Rightarrow \quad y = e.$$

Therefore,

$$\boxed{\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e.}$$

Example 2: Indeterminate Form ∞^0

Compute the limit:

$$\lim_{x\to\infty} x^{1/x}.$$

As $x \to \infty$, we have:

$$x \to \infty, \quad \frac{1}{x} \to 0,$$

so the expression is of the indeterminate form ∞^0.

Let

$$y = x^{1/x}.$$

Taking the natural logarithm,

$$\ln y = \frac{1}{x}\ln x.$$

Now compute the limit:

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0.$$

Thus,

$$\ln y = 0 \quad \Rightarrow \quad y = e^0 = 1.$$

Therefore,

$$\boxed{\lim_{x\to\infty} x^{1/x} = 1.}$$

Example 4: Indeterminate Form $\infty - \infty$

Find the limit

$$\lim_{x\to\infty} \left(\sqrt{x^2+x} - x\right).$$

Let

$$L = \lim_{x\to\infty} \left(\sqrt{x^2+x} - x\right).$$

Multiply by the conjugate:

$$L = \lim_{x\to\infty} \frac{(\sqrt{x^2+x} - x)(\sqrt{x^2+x} + x)}{\sqrt{x^2+x} + x} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x} + x}.$$

Using L'Hôpital's Rule:

$$L = \lim_{x\to\infty} \frac{1}{\frac{2x+1}{2\sqrt{x^2+x}} + 1} = \frac{1}{2}.$$

Example 5: Indeterminate Form 0^0

Find the limit

$$\lim_{x\to 0^+} x^x.$$

Let

$$y = x^x.$$

Taking the natural logarithm:

$$\ln y = x \ln x.$$

Using the result from Example 3:

$$\lim_{x \to 0^+} x \ln x = 0.$$

Thus,

$$\ln y \to 0 \quad \Rightarrow \quad y \to e^0 = 1.$$

$$\boxed{\lim_{x \to 0^+} x^x = 1}$$

2.6 limits and Continuity

A function $f(x)$ is continuous at $x = a$ if:

$$\lim_{x \to a} f(x) = f(a)$$

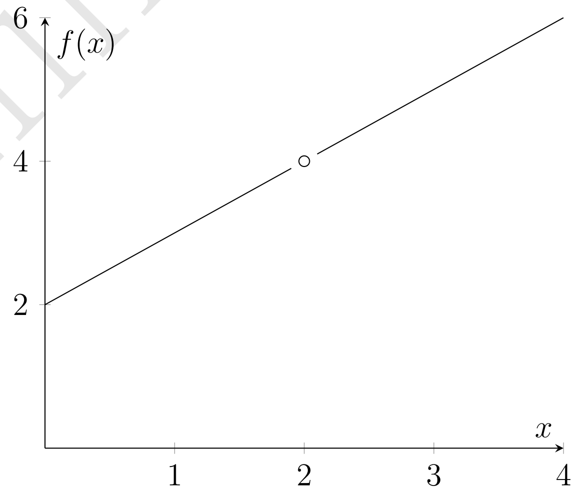
2.6.1 Types of discontinuities

- **Removable(Hole)**: limit exists, function undefined **Example:**

$$f(x) = \frac{x^2 - 4}{x - 2}, \quad x \neq 2$$

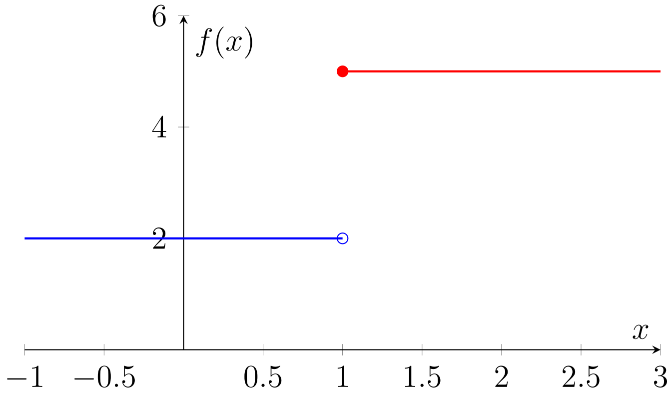
Factor: $f(x) = x + 2, x \neq 2$ Hole at $x = 2$, limit exists: $\lim_{x \to 2} f(x) = 4$

Hole at $x = 2$



- **Jump**: left and right limits exist but are unequal **Example**

$$f(x) = \begin{cases} 2, & x < 1, \\ 5, & x \geq 1. \end{cases}$$



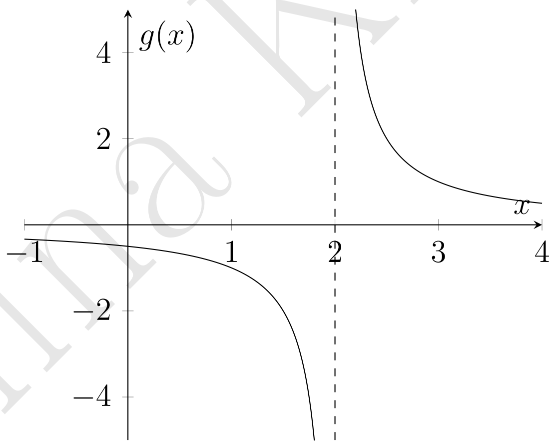
- **Infinite**: vertical asymptote, function goes to $\pm\infty$ near $x = a$.
  A vertical line $x = a$ is a vertical asymptote if:

$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \pm\infty$$

Example:

$$g(x) = \frac{1}{x - 2}$$

Vertical Asymptote at $x = 2$



2.6.2 Horizontal Asymptotes

Definition A horizontal line $y = L$ is a horizontal asymptote if:

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$

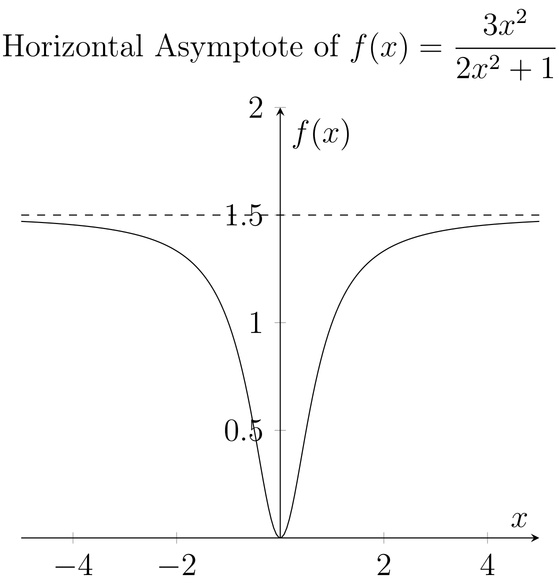
Example

$$f(x) = \frac{3x^2}{2x^2 + 1}.$$

The horizontal asymptote is:

$$y = \frac{3}{2}.$$

Graph of Horizontal Asymptote



2.6.3 Oblique Asymptote

Definition. A line $y = ax + b$ is called an *oblique asymptote* of a function $f(x)$ if

$$\lim_{x \to \pm\infty} \big(f(x) - (ax+b)\big) = 0.$$

For a rational function

$$f(x) = \frac{P(x)}{Q(x)},$$

an oblique asymptote exists when

$$\deg(P) = \deg(Q) + 1,$$

and the asymptote is obtained by polynomial division.

Example

Consider the function:

$$f(x) = \frac{x^2+1}{x}.$$

By polynomial division:

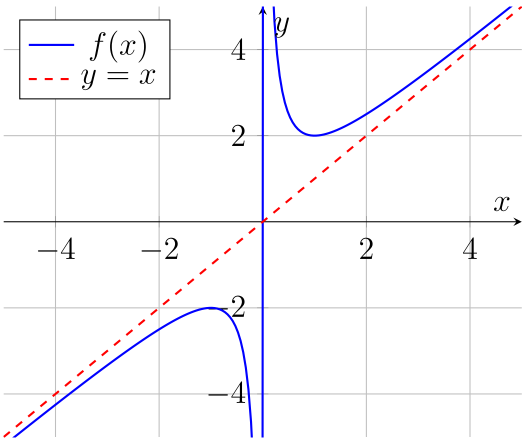
$$f(x) = x + \frac{1}{x}.$$

Since

$$\lim_{x \to \pm\infty} \frac{1}{x} = 0,$$

the oblique asymptote is:

$$y = x.$$



2.6.4 Derivatives

A **derivative** represents the rate at which a function changes with respect to its variable.

$$f'(x), \quad \frac{df}{dx}, \quad \frac{dy}{dx}$$

Reading derivatives:

Expression	Reading
$f'(x)$	f prime of x
$f''(x)$	f double prime of x
$f^{(3)}(x)$	f triple prime of x or f to the third derivative
$\frac{df}{dx}$	the derivative of f with respect to x
$\frac{d^2y}{dx^2}$	the second derivative of y with respect to x
$\frac{d}{dx}(x^3)$	the derivative with respect to x of x cubed

Examples:

$$f(x) = x^2 \quad \Rightarrow \quad f'(x) = 2x$$

Reading: f of x equals x squared, therefore f prime of x equals two x.

$$\frac{d}{dx}(e^x) = e^x$$

Reading: the derivative with respect to x of e to the x equals e to the x.

Derivative Rules

Derivative of a Constant A constant never changes, and so its derivative is always zero:

$$c' = 0$$

Sum Rule: The derivative of the sum of two functions equals the sum of their individual derivatives.

$$(f + g)' = f' + g'$$

Difference Rule: The derivative of the difference of two functions equals the difference of their individual derivatives.

$$(f - g)' = f' - g'$$

Constant Multiple Rule: The derivative of a constant multiplied by a function equals the constant multiplied by the derivative of the function.

$$(kf)' = kf'$$

Product Rule: The derivative of the product of two functions equals the first function multiplied by the derivative of the second, plus the second function multiplied by the derivative of the first.

$$(fg)' = f'g + fg'$$

Quotient Rule: The derivative of the quotient of two functions equals the derivative of the numerator multiplied by the denominator, minus the numerator multiplied by the derivative of the denominator, all divided by the square of the denominator.

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}, \qquad g \neq 0$$

Power Rule: When a function is raised to a power, its derivative reflects both the power and the inner change:

$$((f(x))^n)' = n\,(f(x))^{n-1} f'(x).$$

Chain Rule: The derivative of a composite function equals the derivative of the outer function evaluated at the inner function multiplied by the derivative of the inner function.

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

Leibniz Rule (Derivative of a Product of Several Functions)

The derivative of a product of several functions is obtained by differentiating one function at a time while keeping the others unchanged, then summing all these terms.

$$\frac{d}{dx}[u(x)v(x)w(x)] = u'(x)v(x)w(x) + u(x)v'(x)w(x) + u(x)v(x)w'(x)$$

More generally, for n functions $f_1(x), f_2(x), \ldots, f_n(x)$:

$$\frac{d}{dx}\Big[f_1(x)f_2(x)\cdots f_n(x)\Big] = \sum_{k=1}^{n} \Big(f_1(x)\cdots f_{k-1}(x)f_k'(x)f_{k+1}(x)\cdots f_n(x)\Big)$$

2.6.5 Partial Derivatives

When a function depends on more than one variable, *partial derivatives* measure the rate of change with respect to one variable while holding the others fixed.

Notation:

$$\frac{\partial f}{\partial x}, \quad f_x, \quad \frac{\partial^2 f}{\partial x^2}, \quad f_{xy}$$

Reading partial derivatives:

Expression	Reading
$\dfrac{\partial f}{\partial x}$ / f_x	partial derivative of f with respect to x / f sub x
$\dfrac{\partial^2 f}{\partial x^2}$ / f_{xx}	the second partial derivative of f with respect to x / f sub xx
$\dfrac{\partial f}{\partial x}/f_{xy}$	the second partial derivative of f with respect to x and y /f sub xy

Examples:
Let $f(x, y) = x^2y + 3xy^2$. Then

$$\frac{\partial f}{\partial x} = 2xy + 3y^2, \qquad \frac{\partial f}{\partial y} = x^2 + 6xy.$$

Reading: the partial derivative of f with respect to x equals two x y plus three y squared; the partial derivative of f with respect to y equals x squared plus six x y.

Clairaut's Theorem (symmetry of mixed partials): if f and its partial derivatives of order two are continuous near a point, then

$$f_{xy} = f_{yx}.$$

Reading examples

- $\dfrac{\partial}{\partial x}(\sin(xy))$ is read: partial derivative with respect to x of sine of x y.
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Chapter 3

Applications of Derivation in Mathematics

3.1 Introduction

Differentiation is one of the fundamental tools of modern mathematics. Beyond its interpretation as a rate of change, the derivative provides:

- Local linear approximation,
- Stability information,
- Optimization criteria,
- Foundations for numerical algorithms,
- Analytical tools for nonlinear problems.

This chapter presents the principal applications of derivation with emphasis on analysis and numerical methods.

3.2 Local Analysis of Functions

Definition 5. *Let $f : \mathbb{R} \to \mathbb{R}$. The derivative of f at x_0 is defined by*

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

Theorem 6 (Local Linearization)**.** *If f is differentiable at x_0, then*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(|x - x_0|).$$

Example 7. *Approximate $\sqrt{4.1}$ using linearization at $x_0 = 4$.*

Let $f(x) = \sqrt{x}$. Then

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad f'(4) = \frac{1}{4}.$$

Thus,

$$\sqrt{4.1} \approx 2 + \frac{1}{4}(0.1) = 2.025.$$

3.3 Optimization Theory

Definition 8. *A point x_0 is called a critical point of f if*

$$f'(x_0) = 0.$$

Theorem 9 (Second Derivative Test)**.** *Let $f \in C^2$ and suppose $f'(x_0) = 0$.*

- *If $f''(x_0) > 0$, then x_0 is a local minimum.*
- *If $f''(x_0) < 0$, then x_0 is a local maximum.*

Example 10. *Consider $f(x) = x^3 - 3x$.*

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1).$$

Critical points: $x = \pm 1$.

$$f''(x) = 6x.$$

- *$f''(1) = 6 > 0 \to$ local minimum.*
- *$f''(-1) = -6 < 0 \to$ local maximum.*

3.4 Taylor Expansion and Error Analysis

Theorem 11 (Taylor Formula with Lagrange Remainder)**.** *Let $f \in C^{n+1}$. Then*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1},$$

for some ξ between a and x.

Example 12. *Approximate $e^{0.2}$ using the second-order Taylor polynomial at 0.*

$$e^x \approx 1 + x + \frac{x^2}{2}.$$

Thus,

$$e^{0.2} \approx 1 + 0.2 + \frac{0.04}{2} = 1.22.$$

3.5 Newton's Method

Consider the nonlinear equation

$$f(x) = 0.$$

Using first-order Taylor expansion:

$$f(x) \approx f(x_n) + f'(x_n)(x - x_n).$$

Setting this approximation equal to zero yields:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Theorem 13 (Quadratic Convergence)**.** *Suppose:*

- $f \in C^2$,
- $f(\alpha) = 0$,
- $f'(\alpha) \neq 0$.

Then Newton's method converges quadratically to α for initial values sufficiently close to α.

Example 14. *Solve $x^2 - 2 = 0$.*

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

Starting from $x_0 = 1$:

$$x_1 = 1.5, \quad x_2 = 1.4167, \quad x_3 = 1.4142.$$

3.6 Multivariable Derivatives

Let $f : \mathbb{R}^n \to \mathbb{R}$.

Definition 15. *The gradient of f is defined by*

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

Necessary condition for extremum:

$$\nabla f(x_0) = 0.$$

Definition 16. *The Hessian matrix of f is*

$$H_f(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right).$$

The Hessian is used to classify critical points and derive multidimensional Newton methods.

3.7 Applications to Differential Equations

Consider

$$y' = f(x, y).$$

Linearization near an equilibrium y_0:

$$y' \approx f_y(y_0)(y - y_0).$$

This determines the stability of the equilibrium.

3.8 Numerical Differentiation

Forward difference:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$

Central difference:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}.$$

Using Taylor expansion:

- Forward difference: order $O(h)$,
- Central difference: order $O(h^2)$.

3.9 Conclusion

Derivation plays a central role in:

- Local analysis of functions,
- Optimization theory,
- Nonlinear equation solving,
- Differential equations,
- Numerical approximation,
- Stability analysis.

In applied mathematics, differentiation governs convergence, error behavior, and structural properties of mathematical models.

3.10 Exercises

Exercise 17. *Let* $f(x) = \ln(1+x)$.

a) Compute the second-order Taylor polynomial of f *at* 0.

b) Use it to approximate $\ln(1.1)$.

Solution 18. ***a)*** *We compute derivatives:*

$$f(x) = \ln(1+x), \quad f'(x) = \frac{1}{1+x}, \quad f''(x) = -\frac{1}{(1+x)^2}.$$

At $x = 0$*:*

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -1.$$

Thus the second-order Taylor polynomial is

$$P_2(x) = x - \frac{x^2}{2}.$$

b) *For $x = 0.1$:*

$$\ln(1.1) \approx 0.1 - \frac{0.01}{2} = 0.095.$$

Exact value: 0.09531 *(small error).*

Exercise 19. *Consider $f(x) = x^3 - 6x^2 + 9x$.*

a) Find all critical points.

b) Determine their nature using the second derivative test.

Solution 20.

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3).$$

Critical points: $x = 1$ and $x = 3$.
Second derivative:

$$f''(x) = 6x - 12.$$

$$f''(1) = -6 < 0 \Rightarrow \textit{local maximum.}$$

$$f''(3) = 6 > 0 \Rightarrow \textit{local minimum.}$$

Exercise 21. *Apply two iterations of Newton's method to approximate a root of*

$$f(x) = x^3 - 2x - 5,$$

starting from $x_0 = 2$.

Solution 22.

$$f'(x) = 3x^2 - 2.$$

Newton iteration:

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}.$$

For $x_0 = 2$:

$$x_1 = 2 - \frac{8 - 4 - 5}{12 - 2} = 2 - \frac{-1}{10} = 2.1.$$

Now compute x_2:

$$x_2 = 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2}.$$

$$(2.1)^3 = 9.261, \quad 3(2.1)^2 = 13.23.$$

$$x_2 \approx 2.1 - \frac{9.261 - 4.2 - 5}{13.23 - 2} = 2.1 - \frac{0.061}{11.23} \approx 2.0946.$$

Thus the approximation after two iterations is

$$x \approx 2.0946.$$

Exercise 23. *Let* $f(x, y) = x^2 + y^2 - 4x - 6y$.

a) *Find the critical point.*

b) *Determine its nature using the Hessian matrix.*

Solution 24. *Compute the gradient:*

$$\nabla f = (2x - 4,\ 2y - 6).$$

Set equal to zero:

$$2x - 4 = 0 \Rightarrow x = 2,$$

$$2y - 6 = 0 \Rightarrow y = 3.$$

Critical point: $(2, 3)$.
Hessian matrix:

$$H = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Since the Hessian is positive definite, the critical point is a local minimum.

Exercise 25. *Show using Taylor expansion that the central difference formula*

$$\frac{f(x+h) - f(x-h)}{2h}$$

is of order $O(h^2)$.

Solution 26. *Using Taylor expansion:*

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f^{(3)}(\xi_1),$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f^{(3)}(\xi_2).$$

Subtracting:

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{6}\left(f^{(3)}(\xi_1) + f^{(3)}(\xi_2)\right).$$

Dividing by $2h$*:*

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + O(h^2).$$

Thus the method has second-order accuracy.

3.10.1 Integral Operator

An **integral** represents the area under a curve or the accumulation of quantities.

Indefinite Integrals

$$\int f(x)\,dx$$

Reading: the integral of f of x with respect to x.

Expression	Reading
$\int x^2\,dx$	the integral of x squared with respect to x
$\int e^x\,dx$	the integral of e to the x with respect to x
$\int (3x+1)\,dx$	the integral of three x plus one with respect to x

Examples:

$$\int x^2\,dx = \frac{x^3}{3} + C$$

Reading: the integral of x squared with respect to x equals x cubed over three plus c.

Definite Integrals

$$\int_a^b f(x)\,dx$$

Reading: the integral from a to b of f of x with respect to x.

Expression	Reading
$\int_0^1 x\,dx$	the integral from zero to one of x with respect to x
$\int_2^5 (x^2+1)\,dx$	the integral from two to five of x squared plus one with respect to x
$\int_{-\pi}^{\pi} \sin x\,dx$	the integral from minus pi to pi of sine x with respect to x

Examples:

$$\int_0^1 x\,dx = \frac{1}{2}$$

Reading: the integral from zero to one of x with respect to x equals one half.

3.11 Average Function Value

The first application of integrals that we consider is the **average value of a function**.

Definition

The average value of a continuous function $f(x)$ on the interval $[a, b]$ is given by:

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example 1

Determine the average value of each function.

(a) $f(t) = t^2 - 5t + 6\cos(\pi t)$ **on** $[-1, \frac{5}{2}]$

Solution:

$$f_{\text{avg}} = \frac{1}{\frac{5}{2} - (-1)} \int_{-1}^{5/2} \left(t^2 - 5t + 6\cos(\pi t)\right) dt$$

$$= \frac{1}{\frac{7}{2}} \int_{-1}^{5/2} \left(t^2 - 5t + 6\cos(\pi t)\right) dt = \frac{2}{7} \int_{-1}^{5/2} \left(t^2 - 5t + 6\cos(\pi t)\right) dt$$

Compute the integral term by term:

$$\int t^2 dt = \frac{t^3}{3}, \quad \int -5t dt = -\frac{5t^2}{2}, \quad \int 6\cos(\pi t) dt = \frac{6}{\pi}\sin(\pi t)$$

So,

$$\int_{-1}^{5/2} f(t)\, dt = \left[\frac{t^3}{3} - \frac{5t^2}{2} + \frac{6}{\pi}\sin(\pi t)\right]_{-1}^{5/2}$$

Evaluate at the bounds and substitute to obtain f_{avg}.

(b) $R(z) = \sin(2z)e^{1-\cos(2z)}$ **on** $[-\pi, \pi]$

Solution:

$$R_{\text{avg}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(2z)e^{1-\cos(2z)}\, dz$$

Observe that:

$$\frac{d}{dz}\left(e^{1-\cos(2z)}\right) = 2\sin(2z)e^{1-\cos(2z)}$$

Thus,

$$\sin(2z)e^{1-\cos(2z)} = \frac{1}{2}\frac{d}{dz}\left(e^{1-\cos(2z)}\right)$$

Therefore,

$$\int_{-\pi}^{\pi} \sin(2z)e^{1-\cos(2z)} dz = \frac{1}{2}\left[e^{1-\cos(2z)}\right]_{-\pi}^{\pi}$$

Since:

$$\cos(2\pi) = \cos(-2\pi) = 1 \Rightarrow e^{1-1} = 1$$

We get:

$$\int_{-\pi}^{\pi} \sin(2z)e^{1-\cos(2z)} dz = 0$$

Hence:

$$R_{\text{avg}} = 0$$

Mean Value Theorem for Integrals

If $f(x)$ is continuous on $[a, b]$, then there exists a number $c \in [a, b]$ such that:

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Equivalently,

$$f_{\text{avg}} = f(c)$$

This means the function attains its average value at some point in the interval.

Example 2

Find c such that the Mean Value Theorem holds for:

$$f(x) = x^2 + 3x + 2 \quad \text{on } [1, 4]$$

Solution:

$$f_{\text{avg}} = \frac{1}{4-1}\int_1^4 (x^2+3x+2)\,dx = \frac{1}{3}\int_1^4 (x^2+3x+2)\,dx$$

$$= \frac{1}{3}\left[\frac{x^3}{3} + \frac{3x^2}{2} + 2x\right]_1^4$$

$$= \frac{1}{3}\left(\left[\frac{64}{3} + 24 + 8\right] - \left[\frac{1}{3} + \frac{3}{2} + 2\right]\right)$$

$$= \frac{55}{6}$$

Now solve:

$$f(c) = \frac{55}{6} \Rightarrow c^2 + 3c + 2 = \frac{55}{6}$$

$$6c^2 + 18c + 12 = 55 \Rightarrow 6c^2 + 18c - 43 = 0$$

$$c = \frac{-18 \pm \sqrt{18^2 + 4 \cdot 6 \cdot 43}}{12} = \frac{-18 \pm \sqrt{1356}}{12}$$

Choose the value of c that lies in $[1, 4]$.

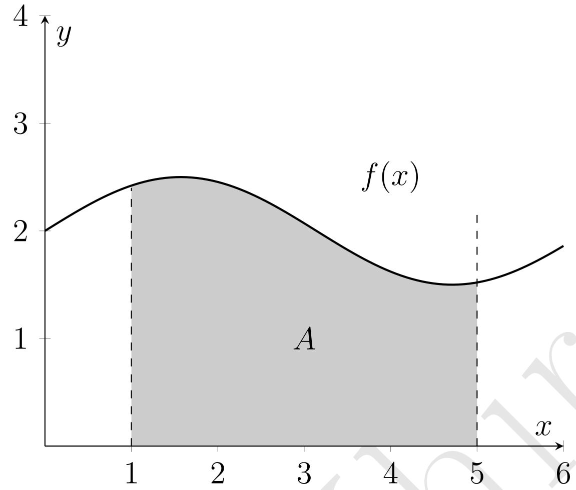
3.12 Areas by Integration

One of the most important applications of definite integrals is the computation of areas.

1. Area Under a Curve

Let $f(x)$ be a continuous function such that $f(x) \geq 0$ on the interval $[a, b]$. The area between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$ is:

$$A = \int_a^b f(x)\, dx$$



Example 1

Find the area under the curve $f(x) = x^2$ on $[0, 2]$.

Solution:

$$A = \int_0^2 x^2\, dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

$$\boxed{A = \frac{8}{3}}$$

2. Area Between a Curve and the x-Axis (General Case)

If $f(x)$ takes both positive and negative values, then the area is:

$$A = \int_a^b |f(x)|\, dx$$

Example 2

Find the area between $f(x) = x$ and the x-axis on $[-1, 1]$.

Solution:

The function is negative on $[-1, 0]$ and positive on $[0, 1]$.

$$A = \int_{-1}^0 (-x)\, dx + \int_0^1 x\, dx$$

$$= \left[\frac{-x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1} = \frac{1}{2} + \frac{1}{2} = 1$$

$$\boxed{A = 1}$$

3. Area Between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions on $[a, b]$ such that $f(x) \geq g(x)$. The area between the curves is:

$$A = \int_a^b [f(x) - g(x)]\, dx$$

Example 3

Find the area between $y = x^2$ and $y = x$ on $[0, 1]$.

Solution:

Since $x \geq x^2$ on $[0, 1]$, we have:

$$A = \int_0^1 (x - x^2)\, dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$\boxed{A = \frac{1}{6}}$$

4. Area with Respect to y

Sometimes it is easier to integrate with respect to y:

$$A = \int_c^d [x_{\text{right}} - x_{\text{left}}]\, dy$$

Example 4

Find the area between $x = y^2$ and $x = y + 2$ for $y \in [0, 1]$.

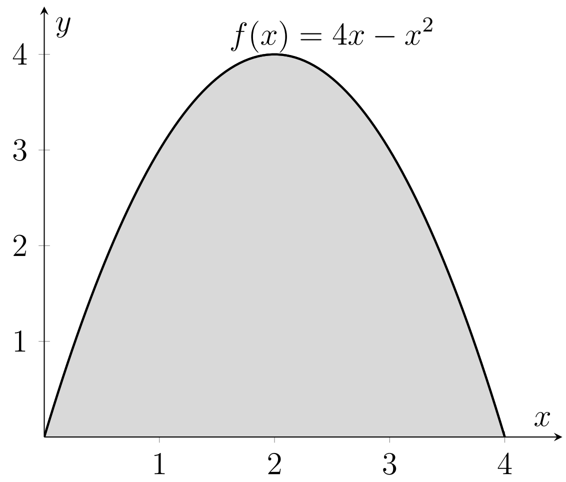
Solution:

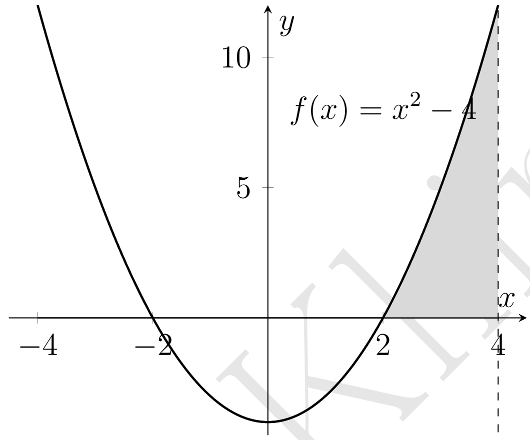
$$A = \int_0^1 \left[(y + 2) - y^2\right] dy$$

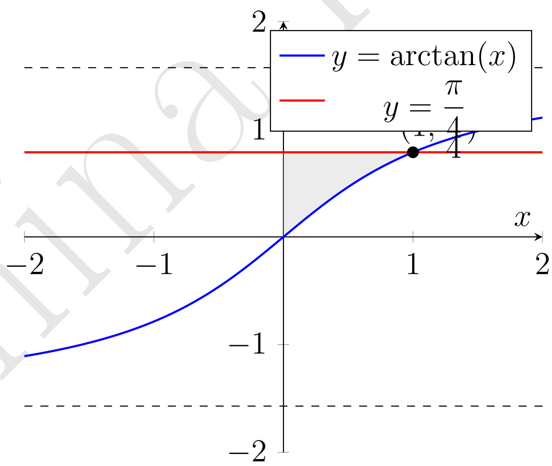
$$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_0^1 = \frac{1}{2} + 2 - \frac{1}{3} = \frac{13}{6}$$

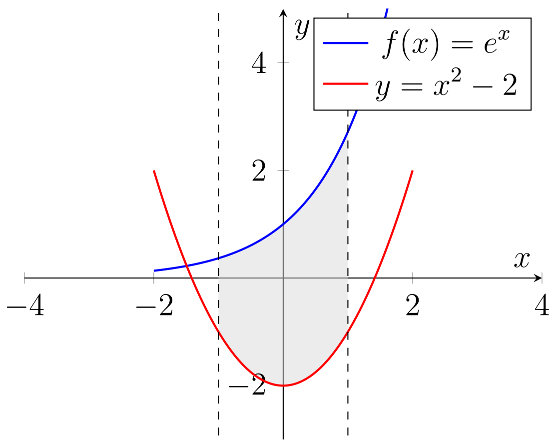
$$\boxed{A = \frac{13}{6}}$$

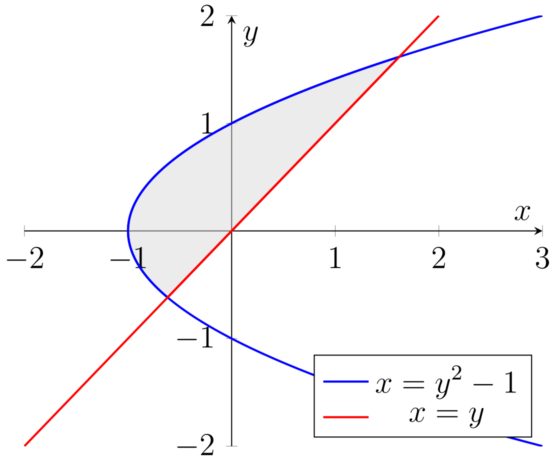
Exercise Calculate the shaded area in each of the following cases

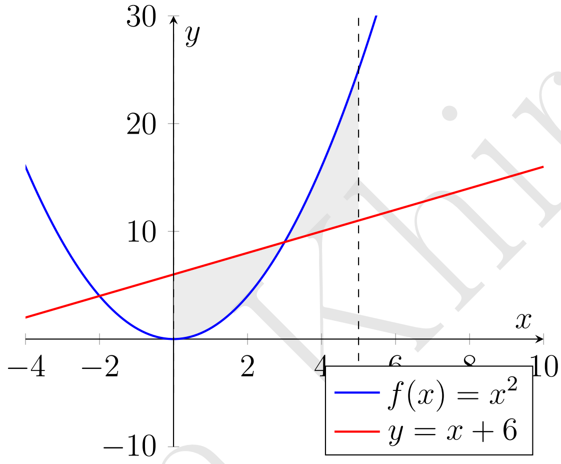

$f(x) = 4x - x^2$
y
x
1
2
3
4


$f(x) = x^2 - 4$
y
x
10
5
−4
−2
2
4


$y = \arctan(x)$
$y = \dfrac{\pi}{4}$
$(1, \frac{\pi}{4})$
x
2
1
−1
−2
−2
−1
1
2


$f(x) = e^x$
$y = x^2 - 2$
y
x
4
2
−2
−4
−2
2
4





Step 1: Find the boundaries

Set $f(x) = 0$ to determine the x-intercepts:

$$4x - x^2 = 0 \quad \Rightarrow \quad x(4 - x) = 0 \quad \Rightarrow \quad x = 0 \text{ or } x = 4$$

Thus, the boundaries are:

$$a = 0, \quad b = 4$$

—

Step 2: Set up the integral

The area under the curve is:

$$A = \int_0^4 (4x - x^2)\, dx$$

—

Step 3: Solve the integral

$$
\begin{aligned}
A &= \int_0^4 4x\,dx - \int_0^4 x^2\,dx \\
&= \left[2x^2\right]_0^4 - \left[\frac{x^3}{3}\right]_0^4 \\
&= (2 \cdot 16 - 0) - \frac{64 - 0}{3} \\
&= 32 - \frac{64}{3} \\
&= \frac{96 - 64}{3} \\
&= \frac{32}{3}
\end{aligned}
$$

—

Answer:
The area in the first quadrant under the curve $f(x) = 4x - x^2$ is:

$$\boxed{\frac{32}{3} \text{ square units}}$$

Step 1: Find the boundaries
The interval is given: $a = 2$, $b = 4$.

—

Step 2: Set up the integral
The area under the curve is:

$$A = \int_2^4 (x^2 - 4)\,dx$$

—

Step 3: Solve the integral

$$
\begin{aligned}
A &= \int_2^4 x^2\,dx - \int_2^4 4\,dx \\
&= \left[\frac{x^3}{3}\right]_2^4 - [4x]_2^4 \\
&= \left(\frac{64}{3} - \frac{8}{3}\right) - (16 - 8) \\
&= \frac{56}{3} - 8 \\
&= \frac{56}{3} - \frac{24}{3} \\
&= \frac{32}{3}
\end{aligned}
$$

—

Answer:

$$\boxed{A = \frac{32}{3} \text{ square units}}$$

Area Using Horizontal Integration

Since it is easier to integrate $x = \tan(y)$ than $y = \arctan(x)$, we rewrite the function in terms of y and use horizontal strips.

$$y = \arctan(x) \quad \Longrightarrow \quad x = \tan(y)$$

So the function becomes:

$$x = g(y) = \tan(y)$$

Bounds

The lower bound is:

$$y = 0 \quad (\text{since } \arctan(x) = 0 \Rightarrow x = 0)$$

The upper bound is:

$$y = \frac{\pi}{4}$$

Area Formula

Using horizontal integration:

$$A = \int_0^{\frac{\pi}{4}} \tan(y)\, dy$$

Computation

$$\int \tan(y)\, dy = -\ln|\cos(y)|$$

$$A = \int_0^{\frac{\pi}{4}} \tan(y)\, dy = [-\ln(\cos y)]_0^{\frac{\pi}{4}}$$

Evaluation

$$A = -\ln\left(\cos\frac{\pi}{4}\right) + \ln(\cos 0)$$

$$A = -\ln\left(\frac{\sqrt{2}}{2}\right) + \ln(1)$$

$$A = \ln\left(\frac{2}{\sqrt{2}}\right) = \ln(\sqrt{2})$$

$$A = \frac{1}{2}\ln(2)$$

$$\boxed{A = \frac{1}{2}\ln(2)}$$

Area between $y = \arctan(x)$ and $y = \dfrac{\pi}{4}$

Intersection:

$$\arctan(x) = \frac{\pi}{4} \Rightarrow x = 1$$

Area:

$$A = \int_0^1 \left(\frac{\pi}{4} - \arctan(x)\right) dx$$

Using:

$$\int \arctan(x)\, dx = x\arctan(x) - \frac{1}{2}\ln(1 + x^2)$$

$$A = \frac{\pi}{4} - \left[x\arctan(x) - \frac{1}{2}\ln(1 + x^2)\right]_0^1$$

$$A = \frac{\pi}{4} - \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right)$$

$$A = \frac{1}{2}\ln 2$$

$$\boxed{A = \frac{1}{2}\ln 2}$$

Area for $x > 2$

We consider the function:

$$f(x) = x^2 - 4$$

The area above the x-axis for $x > 2$ (up to $x = 4$) is:

$$A = \int_2^4 (x^2 - 4)\, dx$$

Compute the integral:

$$\int (x^2 - 4)\, dx = \frac{x^3}{3} - 4x$$

Evaluate:

$$A = \left[\frac{x^3}{3} - 4x\right]_2^4$$

$$A = \left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right)$$

$$A = \frac{16}{3} - \left(-\frac{16}{3}\right)$$

$$A = \frac{32}{3}$$

$$\boxed{A = \frac{32}{3}}$$

Area Between Two Curves

In this case, it is convenient to integrate with respect to x.

Given Functions

$$y = e^x \quad \text{and} \quad y = x^2 - 1$$

Bounds

$$a = -1, \quad b = 1$$

On the interval $[-1, 1]$, the function $y = e^x$ lies above $y = x^2 - 1$.

Area Formula

$$A = \int_{-1}^{1} \big(e^x - (x^2 - 1)\big)\, dx$$

Computation

$$A = \int_{-1}^{1} e^x\, dx - \int_{-1}^{1} (x^2 - 1)\, dx$$

$$= [e^x]_{-1}^{1} - \left[\frac{x^3}{3} - x\right]_{-1}^{1}$$

Evaluation

$$[e^x]_{-1}^{1} = e - \frac{1}{e}$$

$$\left[\frac{x^3}{3} - x\right]_{-1}^{1} = \left(\frac{1}{3} - 1\right) - \left(-\frac{1}{3} + 1\right) = -\frac{2}{3} - \frac{2}{3} = -\frac{4}{3}$$

Final Result

$$A = \left(e - \frac{1}{e}\right) - \left(-\frac{4}{3}\right)$$

$$A = e - \frac{1}{e} + \frac{4}{3}$$

$$\boxed{A = e - \frac{1}{e} + \frac{4}{3}}$$

Area Between Two Curves (Integration with respect to y)

Given Functions

$$x = y^2 - 2 \quad \text{and} \quad x = y$$

Intersection Points

Solve:

$$y^2 - 2 = y$$

$$y^2 - y - 2 = 0$$

$$(y-2)(y+1) = 0$$

$$y = 2 \quad \text{or} \quad y = -1$$

So the bounds are:

$$c = -1, \quad d = 2$$

Left and Right Functions

$$\text{Left: } x = y^2 - 2, \qquad \text{Right: } x = 2y$$

Area Formula

$$A = \int_{-1}^{2} \big(y - (y^2 - 2)\big)\, dy$$

Computation

$$A = \int_{-1}^{2} (-y^2 + y + 2)\, dy$$

$$= \left[-\frac{y^3}{3} + y^2 + 2y \right]_{-1}^{2}$$

Evaluation

At $y = 2$:

$$-\frac{8}{3} + 4 + 4 = \frac{16}{3}$$

At $y = -1$:

$$-\left(-\frac{1}{3}\right) + 1 - 2 = \frac{1}{3} - 1 = -\frac{2}{3}$$

$$A = \frac{16}{3} - \left(-\frac{2}{3}\right) = \frac{18}{3} = 6$$

Final Answer

$$\boxed{A = 6}$$

Area Between Intersecting Curves

In this case, the curves intersect within the interval, so we must split the area into two integrals.

Given Functions

$$y = x^2, \quad y = x + 6$$

Intersection Point

Solve:

$$x^2 = x + 6$$

$$x^2 - x - 6 = 0$$

$$(x - 3)(x + 2) = 0$$

$$x = 3 \quad \text{or} \quad x = -2$$

Since the region is between $x = 0$ and $x = 5$, we take:

$$x = 3$$

Area Setup

From 0 to 3: the line is above the parabola From 3 to 5: the parabola is above the line

$$A = \int_0^3 \left[(x + 6) - x^2\right] dx + \int_3^5 \left[x^2 - (x + 6)\right] dx$$

Computation

$$A = \int_0^3 (-x^2 + x + 6)\, dx + \int_3^5 (x^2 - x - 6)\, dx$$

$$= \left[-\frac{x^3}{3} + \frac{x^2}{2} + 6x\right]_0^3 + \left[\frac{x^3}{3} - \frac{x^2}{2} - 6x\right]_3^5$$

Evaluation

First integral:

$$= \left(-9 + \frac{9}{2} + 18\right) = \frac{27}{2}$$

Second integral:

$$= \left(\frac{125}{3} - \frac{25}{2} - 30\right) - \left(\frac{27}{3} - \frac{9}{2} - 18\right)$$

$$= \frac{125}{3} - \frac{25}{2} - 30 - 9 + \frac{9}{2} + 18$$

$$= \frac{125}{3} - 21 = \frac{62}{3}$$

Final Result

$$A = \frac{27}{2} + \frac{62}{3} = \frac{81 + 124}{6} = \frac{205}{6}$$

$$\boxed{A = \frac{205}{6}}$$

Arc Length of a Curve

Let f be a smooth function over the interval $[a, b]$. Then the arc length of the portion of the graph of $y = f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Remark: A function is called *smooth* if it is continuous and has a continuous derivative on the interval $[a, b]$.

Example: Arc Length of a Curve

Problem: Find the exact length of the curve

$$y = 2x^{3/2}$$

on the interval $[0, 1]$.

Step 1: Recall the formula
The arc length of a curve $y = f(x)$ is:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Step 2: Compute the derivative

$$f(x) = 2x^{3/2}$$

Using the power rule:

$$f'(x) = 2 \cdot \frac{3}{2} x^{1/2} = 3\sqrt{x}$$

Step 3: Substitute into the formula

$$L = \int_0^1 \sqrt{1 + (3\sqrt{x})^2}\, dx$$

$$L = \int_0^1 \sqrt{1 + 9x}\, dx$$

Step 4: Evaluate the integral
Let:

$$u = 1 + 9x \quad \Rightarrow \quad du = 9\, dx$$

$$dx = \frac{du}{9}$$

Change the limits:

$$x = 0 \Rightarrow u = 1, \quad x = 1 \Rightarrow u = 10$$

Thus:

$$L = \int_1^{10} \sqrt{u} \cdot \frac{1}{9}\, du = \frac{1}{9} \int_1^{10} u^{1/2}\, du$$

Step 5: Integrate

$$\int u^{1/2} du = \frac{2}{3} u^{3/2}$$

So:

$$L = \frac{1}{9} \cdot \frac{2}{3} u^{3/2} = \frac{2}{27} u^{3/2}$$

Step 6: Apply the bounds

$$L = \frac{2}{27} \left[u^{3/2}\right]_1^{10} = \frac{2}{27} \left(10^{3/2} - 1\right)$$

Final Answer:

$$\boxed{L = \frac{2}{27} \left(10^{3/2} - 1\right)}$$

Remark:
Since $10^{3/2} = 10\sqrt{10}$, we can also write:

$$L = \frac{2}{27} \left(10\sqrt{10} - 1\right)$$

Exercise

Prove that the arc length of

$$f(x) = \ln(\sin x)$$

on the interval $[\pi/4, \pi/3]$ is

$$L = \ln\left(\frac{\sqrt{2}+1}{\sqrt{3}}\right)$$

Step 1: Arc length formula

For a function $y = f(x)$, the arc length is given by:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Here, $f(x) = \ln(\sin x)$ and the interval is $[\pi/4, \pi/3]$.

Step 2: Compute derivative

$$f'(x) = \frac{d}{dx}\ln(\sin x) = \frac{\cos x}{\sin x} = \cot x$$

Step 3: Substitute into the formula

$$L = \int_{\pi/4}^{\pi/3} \sqrt{1 + \cot^2 x}\, dx$$

Recall the Pythagorean identity:

$$1 + \cot^2 x = \csc^2 x$$

So:

$$L = \int_{\pi/4}^{\pi/3} \csc x\, dx$$

Step 4: Integral of $\csc x$

$$\int \csc x\, dx = \ln\left|\tan\frac{x}{2}\right| + C$$

Thus:

$$L = \left[\ln\left(\tan\frac{x}{2}\right)\right]_{\pi/4}^{\pi/3}$$

Step 5: Evaluate the bounds

$$\tan\frac{\pi}{3}/2 = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}, \quad \tan\frac{\pi}{4}/2 = \tan\frac{\pi}{8} = \sqrt{2} - 1$$

So:

$$L = \ln\left(\frac{1/\sqrt{3}}{\sqrt{2}-1}\right) = \ln\left(\frac{1}{\sqrt{3}(\sqrt{2}-1)}\right)$$

Rationalizing the denominator:

$$\frac{1}{\sqrt{3}(\sqrt{2}-1)} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} = \frac{\sqrt{2}+1}{\sqrt{3}}$$

Step 6: Final Answer

$$\boxed{L = \ln\left(\frac{\sqrt{2}+1}{\sqrt{3}}\right)}$$

3.12.1 Multiple Integrals (Double and Triple Integrals)

Multiple integrals extend the single-variable integral to functions of several variables and compute volumes and accumulated quantities over regions in higher dimensions.

Double integrals

$$\iint_R f(x,y)\,dA$$

Reading: the double integral of f of x,y over the region R with respect to area.

If R is described in Cartesian coordinates, dA is often written as $dx\,dy$ or $dy\,dx$ and we compute by iterated integrals:

$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right) dx$$

Example:

$$\iint_{[0,1]\times[0,2]} (x+2y)\,dA = \int_0^1 \int_0^2 (x+2y)\,dy\,dx.$$

Triple integrals

$$\iiint_V f(x,y,z)\,dV$$

Reading: the triple integral of f of x,y,z over the volume V with respect to volume.

Multiple Integrals — Exercises

Exercise 1 (double integral, iterated):

(1) Evaluate $\iint_R (x+2y)\,dA$ where $R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 2\}$.

Solution:

$$\iint_R (x+2y)\,dA = \int_0^1 \left(\int_0^2 (x+2y)\,dy\right) dx = \int_0^1 \big(2x+4\big)\,dx = \big[x^2+4x\big]_0^1 = 1+4 = 5.$$

Exercise 2 (double integral over triangular region):

(2) Evaluate $\iint_T (x+y)\,dA$ where T is the triangle with vertices $(0,0),(1,0),(1,1)$.

Solution: The triangle T can be described by $0 \le x \le 1$, $0 \le y \le x$. Then

$$\iint_T (x+y)\,dA = \int_0^1 \int_0^x (x+y)\,dy\,dx = \int_0^1 \left[xy + \tfrac{y^2}{2}\right]_{y=0}^{y=x} dx = \int_0^1 \left(x^2 + \tfrac{x^2}{2}\right) dx = \int_0^1 \tfrac{3x^2}{2}\,dx = \left[\tfrac{3x^3}{6}\right]_0^1 =$$

Exercise 3 (triple integral in cylindrical coordinates):

(3) Let V be the cylinder $x^2+y^2 \le 1$, $0 \le z \le 2$. Compute $\iiint_V z\,dV$.

Solution: Use cylindrical coordinates $dV = r\,dr\,d\theta\,dz$ and z independent of r, θ:

$$\iiint_V z\,dV = \int_0^{2\pi} \int_0^1 \int_0^2 z\,r\,dz\,dr\,d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 r\,dr\right)\left(\int_0^2 z\,dz\right) = (2\pi)\left[\tfrac{r^2}{2}\right]_0^1 \left[\tfrac{z^2}{2}\right]_0^2 = 2\pi\cdot\tfrac{1}{2}\cdot 2 =$$

Exercise 4 (change to polar coordinates):

(4) Compute $\iint_D (x^2+y^2)\,dA$ where D is the disk $x^2+y^2 \le 4$.

Solution: In polar coordinates $x^2+y^2 = r^2$, $dA = r\,dr\,d\theta$ and region: $0 \le r \le 2$, $0 \le \theta \le 2\pi$:

$$\iint_D r^2\,dA = \int_0^{2\pi} \int_0^2 r^2 \cdot r\,dr\,d\theta = \int_0^{2\pi} \int_0^2 r^3\,dr\,d\theta = \left(\int_0^{2\pi} d\theta\right)\left[\tfrac{r^4}{4}\right]_0^2 = 2\pi \cdot \tfrac{16}{4} = 8\pi.$$

Further suggested exercises (for practice):

- Compute $\dfrac{\partial^2}{\partial x^2}(\sin(xy))$ and $\dfrac{\partial^2}{\partial x \partial y}(e^{xy})$.
- Evaluate $\iint_R xy\,dA$ where R is the rectangle $[0,2]\times[0,1]$.
- Compute $\iiint_W (x+y+z)\,dV$ where W is the box $0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$.