University of M'Sila,
Faculty of Mathematics and Computer Science / Department of Mathematics
Master of **Functional Analysis S 2**
– ***Analyse de Fourier*** –

**Final Exam** May 2026

** We use the notations of the lesson.
** We take into account of the presentation of the copy.

**Exercise 1. (9 pts.)**(Lesson.) **(1)** Give the definition of :
– $\mathcal{S}(\mathbb{R})$, $\mathcal{S}_\infty(\mathbb{R})$, $\mathcal{S}'_\infty(\mathbb{R})$. – The Fourier transform of a function. – The Fourier transform of a distribution. – The space $W^m(\mathbb{R})$.

**(2)** Prove that $H^s(\mathbb{R}) \subset L_2(\mathbb{R})$ for all $s \geq 0$. – Find $s$ and prove that $H^s(\mathbb{R}) \subset L_\infty(\mathbb{R})$.

**(3)** Complete the missing assertions in the following sentence :

- If $f = g$ in $\mathcal{S}'_\infty(\mathbb{R}) \implies f(x) = \ldots\ldots$ and $\operatorname{supp} \ldots\ldots$

**(4)** Find $f \in \mathcal{S}_3(\mathbb{R})$ such that $f \notin \mathcal{S}_4(\mathbb{R})$.

**(5)** Let $f(x) = 1$ if $x \in [-\frac{1}{2}, \frac{1}{2}]$ and $f(x) = 0$ otherwise.
Calculate $\widehat{f}$. Calculate $\displaystyle\int_0^\infty \frac{\sin t}{t} dt$ and $\displaystyle\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt$.

**Exercise 2. (6 pts.)** **(1)** Let $f_\beta(x) = e^{-\beta|x|}$ and $g(x) = e^{-\beta|x|}\cos(2\pi\beta x)$, $\beta > 0$. Let $h(x) = \arctan x$.

(a) Calculate $\widehat{f_\beta}(\xi)$, $\widehat{g}(\xi)$, $\widehat{h}(\xi)$. (b) Calculate $\displaystyle\int_0^\infty \frac{1}{(1+t^2)^2} dt$.

**(2)** Let $K_{a,b}(x) = \displaystyle\int_{-\infty}^{\infty} e^{-a|x-y|-b|y|} dy$, $a > 0$, $b > 0$. Calculate $\widehat{K_{a,b}}(\xi)$ and deduce directly $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(|x-y|+|y|)}\, dx dy$.

**(3)** Let $\gamma \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}\gamma \subset [-\frac{1}{2}, \frac{1}{2}]$. We put $G(\xi) = (1 + 4\pi^2\xi^2)^2\gamma(\xi)$. Calculate $(\mathcal{F}^{-1}G) * K_{1,1}$.

**Exercise 3. (5 pts.)** **(1)** Let $f(x) = \dfrac{1}{\sqrt{1+x^2}}$. Prove that $f \in H^1(\mathbb{R})$.

**(2)** Let $g(x) = e^{-|x|}$. Find $s \in \mathbb{R}$ such that $g \in H^s(\mathbb{R})$.

**(3)** Let $k(x) = 1$ if $x \in [0, 1]$ and $k(x) = 0$ otherwise.
(a) Prove that $k \in \mathcal{D}'(\mathbb{R})$.
(b) Prove that $H^1(\mathbb{R}) \neq L_2(\mathbb{R})$. Recall that $H^1(\mathbb{R}) \subset L_2(\mathbb{R})$ exercise 1-(2)

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**Scale :**
Ex. 1 : 5 + 1 + 1 + 1 + 1
Ex. 2 : (0.5+1+1.25) + 0.75 + (1.25+0.75) + 0.5
Ex. 3 : 1.5 + 1.5 + (0.5+1.5)

CORRECTION Analyse de Fourier

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**Exercise 1** (1) See the lesson.

(2) • By Perseval

$$\|f\|_2^2 = \|\hat{f}\|_2^2 = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi \leq \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 (1+|\xi|^2)^s d\xi$$

$= \|f\|_{H^s}^2$ (Since $s \geq 0$ $1 \leq (1+|\xi|^2)^s$).

• $$|f(x)| \leq \|\hat{f}\|_1 = \int_{-\infty}^{\infty} |\hat{f}(\xi)| d\xi = \int_{-\infty}^{\infty} |\hat{f}(\xi)| (1+|\xi|^2)^{\frac{s}{2}} (1+|\xi|^2)^{-\frac{s}{2}} d\xi$$

$\leq \|f\|_{H^s} \left(\int_{-\infty}^{\infty} (1+|\xi|^2)^{-s} d\xi\right)^{1/2}$ inequality of Cauchy Schwartz

$$\int_{-\infty}^{\infty} (1+|\xi|^2)^{-s} d\xi = 2 \int_0^1 + \int_1^{\infty} \cdots$$

$\int_1^{\infty} (1+|\xi|^2)^{-s} d\xi \sim \int_1^{\infty} |\xi|^{-2s} d\xi < \infty$ if $-2s+1 < 0$

i.e. $s > 1/2$.

(3) $f = g$ in $\mathcal{S}'_{\infty}(\mathbb{R}) \implies f(x) = g(x) + p(x)$, where $p(x)$ is a polynomial and $\operatorname{supp} \widehat{f-g} = \{0\}$.

(4) Let $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\hat{\varphi}(0) \neq 0$, we put $\underline{\underline{f = \varphi'''}}$.

$\hat{f}(\xi) = (-2i\pi\xi)^3 \hat{\varphi}(\xi)$

we have $\hat{f}(0) = \hat{f}'(0) = \hat{f}''(0) = 0$

and $\hat{f}'''(0) = 6(-2i\pi)^3 \hat{\varphi}(0) \neq 0$.

$f \in S_3(\mathbb{R})$ and $f \notin S_4(\mathbb{R})$.

« Note if $g \in S_4(\mathbb{R}) \implies \hat{g}'''(0) = 0$ »

(5)

- $\hat{f}(\xi) = \int_{-1/2}^{1/2} e^{-2i\pi x\xi}\,dx = -\frac{1}{2i\pi\xi}\left(e^{-i\pi\xi} - e^{i\pi\xi}\right)$ $\boxed{2\,|\,4}$

$$= \frac{\sin(\pi\xi)}{\pi\xi}$$

- $\int_{-\infty}^{\infty} e^{2i\pi x\xi}\frac{\sin\pi\xi}{\pi\xi}\,d\xi = f(x) \Rightarrow f(0) = 2\int_0^\infty \frac{\sin\pi\xi}{\pi\xi}\,d\xi$

$\pi\xi = t \Rightarrow \int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}f(0) = \frac{\pi}{2}$ $\quad (f(0)=1)$.

- Perseval: $\int_{-\infty}^{\infty}\left(\frac{\sin\pi\xi}{\pi\xi}\right)^2 d\xi = 2\int_0^\infty\left(\frac{\sin t}{t}\right)^2\frac{dt}{\pi} = \int_{-\infty}^{\infty}|f(x)|^2dx$

$\Rightarrow \int_0^\infty\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}\int_{-1/2}^{1/2}dx = \frac{\pi}{2}$.

<u>Exercise 2</u> (1) $f_\beta, g \in L_1(\mathbb{R})$ (easy since $\beta>0$).

- $\hat{f}_\beta(\xi) = \int_{-\infty}^{\infty} e^{-2i\pi x\xi - \beta|x|}\,dx = 2\int_0^\infty \cos(2\pi x\xi)\,e^{-\beta x}\,dx$

$$= (\text{by part}) = \cdots = \frac{2\beta}{\beta^2 + 4\pi^2\xi^2}.$$

- $\hat{g}(\xi) = \int_{-\infty}^{\infty} e^{-2i\pi x\xi}\left(\frac{e^{2i\pi\beta x} + e^{-2i\pi\beta x}}{2}\right)e^{-\beta|x|}\,dx$

$= \frac{1}{2}\int_{-\infty}^{\infty}\left(e^{-2i\pi x(\xi-\beta)} + e^{-2i\pi x(\xi+\beta)}\right)e^{-\beta|x|}\,dx$

$= \frac{1}{2}\left(\hat{f}_\beta(\xi-\beta) + \hat{f}_\beta(\xi+\beta)\right)$

$= \frac{\beta}{\beta^2 + 4\pi^2(\xi-\beta)^2} + \frac{\beta}{\beta^2 + 4\pi^2(\xi+\beta)^2}$.

- $h(x) = \arctan x \notin L_1(\mathbb{R})$

We see this in $S'(\mathbb{R})$ since $\arctan x \in L^1_{loc}(\mathbb{R})$

$h'(x) = \frac{1}{1+x^2}$, we use the formula

$$\widehat{h'}(\xi) = 2i\pi\xi\,\hat{h}(\xi).$$

$$h'(x) = \frac{2(2\pi)}{(2\pi)^2+4\pi^2x^2}\cdot\pi = \pi\,\hat{f}_{2\pi}(x) = \pi\,\mathcal{F}^{-1}f_{2\pi}(-x)$$

i.e. $h'(-x) = \pi\,\mathcal{F}^{-1}f_{2\pi}(x) \Rightarrow \widehat{h'}(\xi) = \pi f_{2\pi}(\xi)$

$$2i\pi\xi\,\hat{h}(\xi) = \pi f_{2\pi}(\xi) \Rightarrow \hat{h}(\xi) = \frac{1}{2i\xi}e^{-2\pi|\xi|}.$$

- $\hat{f}_{2\pi}(\xi) = \frac{1}{\pi}\frac{1}{1+\xi^2}$, Perseval one has

$$\int_{-\infty}^{\infty}\left(\frac{1}{1+\xi^2}\right)^2 d\xi = \pi^2\int|\hat{f}(\xi)|^2 = 2\pi^2\int_0^{\infty}e^{-4\pi x}\,dx = \frac{2\pi^2}{4\pi}$$

$$\Rightarrow \int_0^{\infty}\frac{1}{(1+t^2)^2}\,dt = \frac{\pi}{4}.$$

(2) - $K_{a,b}(x) = f_a * f_b(x)$

$$\Rightarrow \widehat{K_{a,b}}(\xi) = \hat{f}_a(\xi)\hat{f}_b(\xi) = \frac{4ab}{(a^2+4\pi^2\xi^2)(b^2+4\pi^2\xi^2)}.$$

- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\pi(|x-y|+|y|)}\,dy\,dx$

$$= \int_{-\infty}^{\infty}K_{\pi,\pi}(x)\,dx = \int_{-\infty}^{\infty}e^{-2i\pi 0.x}K_{\pi,\pi}(x)\,dx$$

$$= \hat{K}_{\pi,\pi}(0) = \frac{4}{\pi^2}$$

(3) $\mathcal{F}(\mathcal{F}^{-1}G * K_{1,1})(\xi) = G(\xi)\hat{K}_{1,1}(\xi) = (1+4\pi^2\xi^2)^2\gamma(\xi)\frac{4}{(1+4\pi^2\xi^2)^2}$

$= 4\gamma(\xi)$

$$\Rightarrow \mathcal{F}^{-1}G * K_{1,1} = \mathcal{F}^{-1}\gamma.$$

## Exercise 3

$\boxed{4/4}$

(1) $\int_{-\infty}^{\infty} |f(x)|^2 dx = \arctan x \Big|_{-\infty}^{\infty} = \pi$

$\int_{-\infty}^{\infty} |f'(x)|^2 dx = \int_{-\infty}^{\infty} 4x^2(1+x^2)^{-4} dx \underset{\infty}{\sim} \int \frac{1}{x^2} dx$

$\Rightarrow \int_{-\infty}^{\infty} |f'(x)|^2 dx < +\infty$

$\|f\|_{H^1} = \|f\|_2 + \|f'\|_2 \Rightarrow f \in H^1(\mathbb{R})$.

Recall that $H^1 \sim W^1$ (see the lesson)

(2) $\hat{g}(\xi) = \hat{f}_1(\xi)$ (exercise 2) $= \frac{2}{1+4\pi^2\xi^2}$

$\|g\|_{H^s}^2 = \int_{-\infty}^{\infty} (1+|\xi|^2)^s \frac{4}{(1+4\pi^2\xi^2)^2} d\xi = 8 \int_0^{\infty} \frac{(1+|\xi|^2)^s d\xi}{(1+4\pi^2\xi^2)^2}$

$= 8\left(\int_0^1 + \int_1^{\infty} \cdots\right)$

$\int_1^{\infty} \cdots \sim \int_1^{\infty} |\xi|^{2s-4} d\xi \Rightarrow 2s-4+1 < 0$ ie. $s < \frac{3}{2}$.

(3) (a) easy. ($k \in L^1_{loc}(\mathbb{R})$)

(b) $H^1 = W^1 \subset L_2$ (exercise 1)

$\int_{-\infty}^{\infty} |k(x)|^2 dx = \int_0^1 dx = 1 \Rightarrow k \in L_2(\mathbb{R})$

$\langle k', \varphi \rangle = -\langle k, \varphi' \rangle = -\int_0^1 \varphi'(x) dx$

$= \varphi(0) - \varphi(1) = \langle \delta_0 - \delta_1 | \varphi \rangle$

$k' = \delta_0 - \delta_1 \notin L_2(\mathbb{R})$

since $k' \notin L^1_{loc}(\mathbb{R})$ recall $L_2(\mathbb{R}) \subset L^1_{loc}(\mathbb{R})$.

$\Rightarrow k \notin H^1(\mathbb{R})$ Hence $H^1(\mathbb{R}) = W^1(\mathbb{R}) \underset{\neq}{\subsetneq} L_2(\mathbb{R})$