University of M'Sila,
Faculty of Mathematics and Computer Science / Department of Mathematics
Master of **Functional Analysis S 2**
– ***Analyse de Fourier*** –

Final Exam May 2025

** We use the notations of the lesson.
** We take into account of the presentation of the copy.

Exercise 1. (9 pts.) (1) (Lesson.) Give the definition of :
– $\mathcal{S}(\mathbb{R})$. – $\mathcal{S}_\infty(\mathbb{R})$. – The Fourier transform of a function. – The Fourier transform of a distribution. – The space $H^s(\mathbb{R})$. – The functions ρ and γ.

(2) Complete the missing assertions in the following sentence :
- If $f = g$ in $\mathcal{S}'_\infty(\mathbb{R}) \Rightarrow f(x) = \ldots\ldots$ and supp ......

(3) Find $f \in \mathcal{S}_3(\mathbb{R})$ and $f \notin \mathcal{S}_4(\mathbb{R})$. Find $g \in \mathcal{S}_\infty(\mathbb{R})$.

(4) Let $f(x,y) = 1$ if $(x,y) \in [-1,1] \times [0,1]$ and $f(x,y) = 0$ otherwise.
Calculate $\widehat{f}$. Calculate $\widehat{f}(\frac{1}{4}, \frac{1}{2})$.

Exercise 2. (6 pts.) (1) Let $a, b \in \mathbb{R}$. Calculate $\delta_a * \delta_b$.

(2) Let $a \in \mathbb{R}$, $f_a(x) = \mathrm{e}^{iax}$, $(i^2 = -1)$, $g(x) = \sin x$ and $h(x) = (\sin x)^2$.
(a) Prove that $f_a, g, h \in \mathcal{S}'(\mathbb{R})$.
(b) Calculate $\widehat{f_a}$, $\widehat{g}$, $\widehat{h}$.

(3) We put $a = 4\pi$. Calculate $(\mathcal{F}^{-1}\gamma) * f_{4\pi}$.

Exercise 3. (5 pts.) (1) Find s such that $\displaystyle\int_0^\infty \frac{1}{(1+t^2)^s}\,\mathrm{d}t$ converges.

(2) Prove and find $\alpha \in \mathbb{R}$ such that $f \in H^{3\alpha}(\mathbb{R}) \Rightarrow \widehat{f} \in L_1(\mathbb{R})$.

(3) Let $k > 0$. Suppose that $\exists c > 0$ such that for all $(\xi, \eta) \in \mathbb{R}^2$:

$$(1+|\eta|^2)^{-k/2} \leq c(1+|\xi-\eta|^2)^{k/2}(1+|\xi|^2)^{-k/2}.$$

Find k' such that : $f \in H^k(\mathbb{R}),\ g \in H^k(\mathbb{R}) \Rightarrow fg \in H^{k'}(\mathbb{R})$.

Scale :
Ex. 1 : 5 + 1 + (1+0.5) + (1+0.5)
Ex. 2 : 1 + [(0.25+0.25+0.25) + (1+1+1.25)] + 1
Ex. 3 : 1 + 1.5 + 2.5

CORRECTION

M1 AF S2 (Analyse de Fourier)

5 Exercise 1. (1) See the lesson.

1 (2) If $f = g$ in $\mathcal{S}'_\infty(\mathbb{R}) \Rightarrow f(x) = g(x) + p(x)$, where $p \in \mathcal{P}_\infty$ and $\operatorname{supp} \mathcal{F}(f-g) = \{0\}$.

1 (3). Let $\varphi \in \mathcal{S}(\mathbb{R})$ s.t. $\hat{\varphi}(0) \neq 0$. $f = \varphi''' \Rightarrow \hat{f}(\xi) = \xi^3 \hat{\varphi}(\xi)$

$$\hat{f}(0) = \hat{f}'(0) = \hat{f}''(0) = 0, \quad \hat{f}'''(0) = 6\hat{\varphi}(0) \neq 0$$

0,5 • let $\psi \in \mathcal{D}(\mathbb{R})$ s.t. $0 \notin \operatorname{supp} \psi$ $(\psi(0) = 0)$

$g = \mathcal{F}^{-1}\psi \Rightarrow \hat{g} = \psi$ $\left(\hat{g}^{(k)}(0) = \psi^{(k)}(0) = 0\right)$.

(4) • $$\hat{f}(\xi,\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2i\pi(x\xi + y\eta)} f(x,y)\, dx\, dy$$

$$= \int_{-1}^{1} e^{-2i\pi x\xi}\, dx \int_0^1 e^{-2i\pi y\eta}\, dy$$

$$= \frac{1}{-2i\pi\xi}\left(e^{-2i\pi\xi} - e^{2i\pi\xi}\right) \frac{1}{-2i\pi\eta}\left(e^{-2i\pi\eta} - 1\right) = \frac{\sin(2\pi\xi)}{\pi\xi} \, \frac{e^{-i\pi\eta}\left(e^{-i\pi\eta} - e^{i\pi\eta}\right)}{-2i\pi\eta}$$

1 $$= e^{-i\pi\eta}\, \frac{\sin(2\pi\xi)\sin(\pi\eta)}{\pi^2 \xi\eta}.$$

0,5 • $$\hat{f}\left(\tfrac{1}{4}, \tfrac{1}{2}\right) = \frac{8}{\pi^2} e^{-i\pi/2} = -\frac{8}{\pi^2} i.$$

1 Exercice 2. (1) $\delta_a * \delta_b = \delta_{a+b}$, indeed:

$$\langle \delta_a * \delta_b \mid \varphi \rangle = \langle \delta_{a,x}, \langle \delta_{b,y} \mid \varphi(x+y)\rangle\rangle, \quad \varphi \in \mathcal{D}(\mathbb{R})$$

$$= \langle \delta_{a,x}, \varphi(x+b)\rangle = \varphi(a+b) = \langle \delta_{a+b} \mid \varphi \rangle.$$

0,25 (2) • $\varphi \in S(\mathbb{R})$

$$\langle f_a | \varphi \rangle = \int f_a(x)\varphi(x)\,dx = \int_{-\infty}^{\infty} e^{iax}\varphi(x)\,dx$$

$$\Rightarrow |\langle f_a | \varphi\rangle| \le \|\varphi\|_1 \quad (\text{i.e. } f \in S'(\mathbb{R}))$$

0,25
0,25 Similarly for g and h.

1 • $\hat{f}_a(\xi) = \delta_{\xi=\frac{a}{2\pi}}$, indeed

$\varphi \in S(\mathbb{R})$: $\langle \hat{f}_a | \varphi\rangle := \langle f_a | \hat{\varphi}\rangle = \int_{-\infty}^{\infty} e^{iax}\hat{\varphi}(x)\,dx$

$$= \int_{-\infty}^{\infty} e^{ia\xi}\hat{\varphi}(\xi)\,d\xi = \varphi(a/2\pi) = \langle \delta_{a/2\pi} | \varphi\rangle.$$

1 • $g(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) = \frac{1}{2i}\left(f_1(x) - f_{-1}(x)\right)$: $\hat{g}(\xi) = \frac{1}{2i}\left(\delta_{\frac{1}{2\pi}} - \delta_{\frac{-1}{2\pi}}\right)$.

0,25 • $h(x) = (g(x))^2$ i.e. $\hat{h} = \hat{g} * \hat{g}$

1 $$\hat{h}(\xi) = \frac{1}{-4}\left(\delta_{1/2\pi} - \delta_{-1/2\pi}\right) * \left(\delta_{1/2\pi} - \delta_{-1/2\pi}\right) = -\frac{1}{4}\left(\delta_{1/\pi} - 2\delta_0 + \delta_{-1/\pi}\right).$$

(3) $\operatorname{supp} \gamma \subset [-3/2, -1/2] \cup [1/2, 3/2]$.

$$\mathcal{F}\left[\mathcal{F}\gamma^{-1} * f_{4\pi}\right](\xi) = \gamma(\xi)\,\hat{f}_{4\pi}(\xi)$$

1 $= \gamma(\xi)\,\delta_{\xi=2} = \gamma(2)\delta_{\xi=2} = 0$ then $\mathcal{F}\gamma^{-1} * f_{4\pi} = 0$.

1 <u>Exercise 3.</u> (1) $\int_0^\infty (1+t^2)^{-s}\,dt$ cv if $s > \frac{1}{2}$, indeed

$$\int_0^\infty (1+t^2)^{-s}\,dt = \left(\int_0^1 + \int_1^\infty\right)(1+t^2)^{-s}\,dt$$

$\int_0^1 \ldots$ converges, since a continuous function...

$\int_1^\infty (1+t^2)^{-s}\,dt < \int_1^\infty t^{-2s}\,dt = \frac{1}{1-2s}\, t^{1-2s}\Big|_1^\infty$ then $1-2s<0$.

(2). $\alpha > \frac{1}{6}$ $f \in H^{3\alpha}(\mathbb{R}) \Rightarrow \hat{f} \in L_1(\mathbb{R})$

indeed:

0,5 $$\int_{\mathbb{R}} |\hat{f}(\xi)| d\xi = \int_{\mathbb{R}} |\hat{f}(\xi)| (1+|\xi|^2)^{3\alpha/2} (1+|\xi|^2)^{-\frac{3\alpha}{2}} d\xi$$

$$\leq \left(\int |\hat{f}(\xi)|^2 (1+|\xi|^2)^{3\alpha} d\xi \right)^{1/2} \left(\int_{\mathbb{R}} (1+|\xi|^2)^{-3\alpha} d\xi \right)^{1/2}$$

1 $$= \|f\|_{H^{3\alpha}} \sqrt{2} \left(\int_0^\infty (1+t^2)^{-3\alpha} dt \right)^{1/2} \quad \text{i.e } 3\alpha > \frac{1}{2}$$

$$\Rightarrow \alpha > 1/6 .$$

(3) $$|\widehat{fg}(\xi)| = |\hat{f} * \hat{g}(\xi)|$$

1 $$\leq \int |\hat{f}(\xi-\eta)| |\hat{g}(\eta)| d\eta$$

$$= \int |\hat{f}(\xi-\eta)| |\hat{g}(\eta)| (1+|\eta|^2)^{+k/2} (1+|\eta|^2)^{-k/2} d\eta$$

$$\leq c (1+|\xi|^2)^{-k/2} \int |\hat{f}(\xi-\eta)| (1+|\xi-\eta|)^{k/2} |\hat{g}(\eta)| (1+|\eta|^2)^{k/2} d\eta$$

0,5 $$\leq c (1+|\xi|^2)^{-k/2} \left(\int |\hat{f}(\xi-\eta)|^2 (1+|\xi-\eta|^2)^k d\eta \right)^{1/2} \text{ Cauchy-Schwarz}$$

$$\times \left(\int |\hat{g}(\eta)|^2 (1+|\eta|^2)^k d\eta \right)^{1/2}$$

$$= c (1+|\xi|^2)^{-k/2} \|f\|_{H^k} \|g\|_{H^k}$$

i.e. $$(1+|\xi|^2)^{k'/2} |\widehat{fg}(\xi)| \leq c \|f\|_{H^k} \|g\|_{H^k} (1+|\xi|^2)^{\frac{k'-k}{2}}$$

then $$\|fg\|_{H^{k'}} \leq c \|f\|_{H^k} \|g\|_{H^k} \left(\int (1+|\xi|^2)^{k'-k} d\xi \right)^{1/2}$$

1 From (1) we put $s = k-k' > 1/2$

$$k' < k - 1/2 .$$

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