University of M'Sila,
Faculty of Mathematics and Computer Science / Department of Mathematics
Master of **Numerical and Mathematical Analysis S 2**
– Distributions Analyse de Fourier II–

Final Exam May 2024

** We use the notations of the lesson.
** We take into account of the presentation of the copy.

Exercise 1. (4 pts.) (1) (Lesson.) Give the definition of :
– $\mathcal{S}(\mathbb{R})$. – $\mathcal{S}'(\mathbb{R})$. – The Fourier transform of a distribution. – The classification of partial differential equations of 2^{ed} order.

Exercise 2. (6 pts.) (1) Let $T = 1 + 2x^2$.
(a) Prove that $T \in \mathcal{S}'(\mathbb{R})$ and calculate $\widehat{T}$.
(b) Is that $\mathcal{F}: \mathcal{S}'(\mathbb{R}) \longrightarrow L^1_{loc}(\mathbb{R})$?

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x) := \begin{cases} 1 - x \text{ if } 0 \leq x < 1. \\ 0 \text{ if } x \geq 1, \text{ even.} \end{cases}$

(a) Calculate $\widehat{f}$ (To be checked). (b) Calculate $\displaystyle\int_0^{\infty} \frac{\cos(y)\sin^2(y)}{y^2}\, dy.$

Exercise 3. (10 pts.) (1) In $\mathbb{R}^n$. we put $r^2 = |x|^2 = x_1^2 + ... + x_n^2$ and $f(x) = g(r)$.
(a) If $f \in C^2(\mathbb{R}^n)$, calculate $\Delta f(x)$ on function of g and r.
(b) Solve the equation $\Delta f = 0$ in $\mathbb{R}^n \setminus \{0\}$.

(2) In $\mathbb{R}^2$. Integer the following differential system

$$\frac{dx}{dt} = -x + \frac{2y^2}{x^2 + y^4}, \quad 2y\frac{dy}{dt} = -y^2 - \frac{2x}{x^2 + y^4}.$$

(3) In $\mathbb{R}^2$. Let the two equations

$$y^2\frac{\partial^2 u}{\partial x^2} + x^2\frac{\partial^2 u}{\partial y^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{xy}\left(y^3\frac{\partial u}{\partial x} + x^3\frac{\partial u}{\partial y}\right) \quad \cdots (E).$$

$$\frac{\partial v}{\partial y} - y\frac{\partial^2 v}{\partial y^2} = 0 \quad \cdots (F).$$

(a) What is the nature of (E) ? (b) Find the solution of (F).

Scale :
Ex. 1 : 4
Ex. 2 : (2+1)+(1.5+1.5)
Ex. 3 : (2+2)+3+(1+2)

CORRECTION May 2024

Distribution Analyse de Fourier II

Exercise 1 • See the lesson.

Exercise 2 (1) $T \in L^1_{Loc}(\mathbb{R}) \Rightarrow T \in S'(\mathbb{R})$

(1) (a) $\langle \hat{T}, \varphi \rangle := \langle T, \hat{\varphi} \rangle \quad , \forall \varphi \in S(\mathbb{R})$

$$= \int_{\mathbb{R}} \hat{\varphi}(\xi) d\xi + 2 \int_{\mathbb{R}} \xi^2 \hat{\varphi}(\xi) d\xi \qquad (x := \xi)$$

$$= \int_{-\infty}^{\infty} e^{2i\pi x \xi} \hat{\varphi}(\xi) d\xi \Big|_{x=0} + \frac{1}{(2i\pi)^2} \left(\int_{-\infty}^{\infty} e^{2i\pi x \xi} \hat{\varphi}(\xi) d\xi \right)'' \Big|_{x=0}$$

$$= \varphi(0) - \frac{1}{4\pi^2} \varphi''(0) = \langle \delta - \frac{1}{4\pi^2} \delta'', \varphi \rangle$$

$$\Rightarrow \hat{T} = \delta - \frac{1}{4\pi^2} \delta''.$$

(b) let $T - 2x^2 =: U$ then $\hat{U} = \delta$

$U \in S'(\mathbb{R}^n)$ and $\delta \notin L^1_{Loc}$

$\mathcal{F} : S'(\mathbb{R}) \longrightarrow L^1_{Loc}(\mathbb{R})$ is false.

(2) (a) • $f(x) = 1 - |x|$ if $|x| \leq 1$ and $f(x) = 0$ otherwise

$\Rightarrow f(x) \in L^1(\mathbb{R}) \qquad \int_{|x| \leq 1} (1 - |x|) dx = 2 \int_0^1 (1 - x) dx = 1.$

• $\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2i\pi x \xi} f(x) dx$

$$= \int_{-\infty}^{\infty} \cos(2\pi x \xi) f(x) dx + \underbrace{i \int_{-\infty}^{\infty} \sin(2\pi x \xi) f(x) dx}_{= 0}$$

$$= 2 \int_0^1 (1 - x) \cos(2\pi x \xi) dx \qquad \text{o d function}$$

$$= \dots = \frac{1}{(\pi \xi)^2} \sin^2(\pi \xi) \quad \text{integration by part.}$$

(b) $f(x) = \int_{-\infty}^{\infty} e^{2i\pi\xi x} \hat{f}(\xi) d\xi$

$= \int_{-\infty}^{\infty} \cos(2\pi x\xi) \hat{f}(\xi) d\xi = 2\int_0^{\infty} \cos(2\pi x\xi) \frac{\sin^2 \pi\xi}{(\pi\xi)^2} d\xi$

$= \frac{2}{\pi} \int_0^{\infty} \cos(2xy) \frac{\sin^2 y}{y^2} dy$

we take $x = \frac{1}{2}$ and $f(\frac{1}{2}) = \frac{1}{2}$ then $\int_0^{\infty} \frac{\cos y \sin^2 y}{y^2} dy = \frac{\pi}{4}$.

Exercise 3 (1) (a) $2r\frac{\partial r}{\partial x_j} = 2x_j$ $\quad j = 1, \dots, n$

$\Rightarrow \frac{\partial r}{\partial x_j} = \frac{x_j}{r}$, then $\frac{\partial^2 r}{\partial x_j^2} = \frac{1}{r} - \frac{x_j^2}{r^3}$.

$\Delta f = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2}$ $\qquad \frac{\partial f}{\partial x_j} = g'(r) \frac{\partial r}{\partial x_j}$

$\frac{\partial^2 f}{\partial x_j^2} = g''(r) \left(\frac{\partial r}{\partial x_j}\right)^2 + g'(r) \frac{\partial^2 r}{\partial x_j^2} = \dots$

$\Delta f = g''(r) + \frac{n-1}{r} g'(r)$.

(b) $\Delta f = 0$ $\quad n = 1$ $\quad g(r) = ar + b$, $(r = |x|)$

$n = 2$ $\quad g(r) = a \log r + b$, $(r = \sqrt{x^2+y^2})$

$n \geq 3$ $\quad g(r) = a r^{2-n} + b$, $(r = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2})$.

(2) $X = x$ and $Y = y^2$ then

$\frac{dX}{dt} = -X + \frac{2Y}{X^2+Y^2}$, $\frac{dY}{dt} = -Y - \frac{2X}{X^2+Y^2}$

then we use polar coordandte $X = r\cos\theta$, $Y = r\sin\theta$

$\frac{dr}{dt} = -r$ and $\frac{d\theta}{dt} = \frac{2}{r^2}$

$r = c_1 e^{-t}$ and $\theta = \frac{1}{c_1} e^{2t} + c_2$.

$r^2 = X^2 + Y^2 = x^2 + y^4$ and $\theta = \operatorname{Arctg} \frac{Y}{X} = \operatorname{Arctg} \frac{y^2}{x}$.

(3) (a) (E) is a parabolic equation (easy).

(b) $y\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y} = 0$

We put $\frac{\partial v}{\partial y}(x,y) = m(x,y)$; $y\frac{\partial m}{\partial y} = m$

i.e $m(x,y) = C_x y \implies v(x,y) = \frac{1}{2}C_x y^2 + D_x$

$v(x,y) = y^2 f(x) + g(x)$, $f, g : \mathbb{R}^2 \to \mathbb{C}$.

We can also cherche an elementary solution of (F) by puting $v(x,y) = h(x)\,k(y)$ ...