Generating functions and combinatorics

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I. Generating functions and combinatorics (another way for counting)
Introduction: In this chapter, we give another way for counting using the generating functions of the sequences of numbers. In first, we start with this introductif example:

Example 1 *Suppose that we have a basket (panier) containing: one apple, one orange, one pear, one banana and one cherry(cerise). In how many ways we can choose two fruits (2 éléments). Evidently, using combinations, the answer is* $\begin{pmatrix} 5 \\ 2 \end{pmatrix} = 10$ *manners. But, we can use the functions of choice of the fruits to give the answer. We abbreviate by: A for an apple, O for an orange, P for a pear, B for a banana and C for the cherry. The function of the choice of the apple is :* $\left(A^0 + A^1\right)$ *(qui veut dire ne pas choisir une pomme, indiqué ici par le* 0 *en exposant **ou bien** choisir une pomme qui est indiqué par le* 1 *dans l'exposant, le choix est désigné par* $+$ *), de même* $\left(O^0 + O^1\right)$ *est la fonction génératrice de l'orange, celle du poire:* $\left(P^0 + P^1\right)$*, celle du banane* $\left(B^0 + B^1\right)$ *et finalement* $\left(C^0 + C^1\right)$ *celle du cerise. Donc la fonction génératrice des différentes choix est :* $\left(A^0 + A^1\right)\left(O^0 + O^1\right)\left(P^0 + P^1\right)\left(B^0 + B^1\right)\left(C^0 + C^1\right)$*. Comme on s'intérèsse au nombres de choix de fruits et non pas à un tel ou tel fruit, on note par* $\left(X^0 + X^1\right) = (1 + X)$ *la fonction choix, où* $1X^1$ *désigne le choix d'un élément de* $\{A, O, P, B, C\}$*. Par suite la fonction génératrice des différentes choix obtenue est* $(1 + X)(1 + X)(1 + X)(1 + X)(1 + X) = (1 + X)^5 = 1 + 5X + 10X^2 + 10X^3 + 5X^4 + X^5$*. Ce polynôme représente bien les différentes choix de fruits du panier contenant les fruits* $\{A, O, P, B, C\}$ *. On a par exemple, une seule façon de ne choisir aucun fruits ce qui est indiqué dans le polynome par :* $1^5 = 1$ *. De meme, on a* 5 *façons de choisir un seul fruit (indiqué par* $5X^1$*), de meme on* $10X^2$ *qui veut dire on a* 10 *façons de choisir* 2 *fruits...etc.*

Remarque: On fait remarquer ici que X ne désigne pas une variable mais une place (placeholder, espace réservé) qui indique seulement en écrivant: $a_n X^n$ qu'on a a_n façons de choisir n éléments.
Algebra of counting
Suppose that we have a set of three objects: $\{a, b, c\}$. From these objects there are three ways to choose one object namely: to choose either a **or** b **or** c. Wich can represented symbolicaly by: $a + b + c$.

Similarly, from these three objects there are three ways to choose two objects:
To choose a **and** b which can be represented by: ab
or b **and** c which can be represented bc,
or c **and** a which can be represented ca.
These choices can be represented by : $ab + bc + ca$.
Finally, from these three objects there is only one way to choose three objects: abc.
If we represent the function of choice of the element a by: $1+aX$ which indicates the two ways of selection are: "not select a" **or** " to select a". The aX here, where X is not a variable but placeholder, indicates only: the choice of one element, namely a). Similarly, $1 + bX$ is the function of choice of the element b and $1 + cX$, the function of choice of c.
The function of different choices from the set $\{a, b, c\}$ is then:

$(1 + aX)(1 + bX)(1 + cX) = 1+aX+bX+cX+abX^2+acX^2+bcX^2+abcX^3$
$= 1 + (a + b + c) X + (ab + ac + bc) X^2 + abcX^3$.

This function indicates, for example, that there is three ways to choose two objects, namely: $\{ab,\ bc,\ ca\}$ and one way to choose the three objects: abc.

Binomial coefficient

Suppose that we have a set of n objects $\{x_1, x_2, ..., x_n\}$ and we wish (on souhaite) to choose a subset of k objects. We either choose x_1 or we don't choose x_1. As before we represent this choice by: $\left(x_1^0 + x_1^1\right)$ or simply by $(1 + x_1)$. We again represent the combination of different choices by the product:

$$(1 + x_1)(1 + x_2)(1 + x_3) ... (1 + x_n)$$

And this product gives:

1
$+\ x_1 + x_2 + ... + x_n$
$+\ x_1x_2 + x_1x_3 + ... + x_{n-1}x_n$
$+x_1x_2x_3 + x_1x_2x_4 + ... + x_{n-2}x_{n-1}x_n$
.
.
.
$+x_1x_2x_3...x_n$.

If we pose $x_1 = x_2 = ... = x_n = x$. Then
$(1 + x_1)(1 + x_2)(1 + x_3) ... (1 + x_n) = (1 + x)^n = a_0+a_1x+a_2x^2+...+a_nx^n$
with $a_k = \begin{pmatrix} n \\ k \end{pmatrix}_{n \geq k \geq 0}$

and we have $(1 + x)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} x^k$. For $x = 1$, we have $2^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix}$.

Generating function of the sequence

Recall that, a sequence from the set A is an ordered list of elements from the set A

$$(a_0, a_1, a_2..., a_n, ...)$$

and this sequence can be finite or infinite. To refer to the infinite sequence $(a_0, a_1, a_2..., a_n, ...)$, we write $(a_n)_{n\in\mathbb{N}}$ or $(a_n)_{n\geq 0}$.

Definition 2 *If A is a set, an infinite sequence of elements of A can be regarded as function $a : \mathbb{N} \longrightarrow A$.*

Pour pouvoir effectuer des opérations algèbriques sur l'ensemble de ces sequences, l'ensemble A est généralement pris comme un anneau, par exemple l'anneau $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$...etc.
A la séquence $(a_0, a_1, a_2, ...a_n, ...)$, nous associons la série formelle $a_0 + a_1X + a_2X^2 + ... + a_nX^n + ...$
On note cette correspondance par:

$$(a_0, a_1, a_2..., a_n, ...) \longleftrightarrow\ a_0 + a_1X + a_2X^2 + ... + a_nX^n + ... = \sum_{n\geq 0} a_nX^n.$$

Dans cette approche, les règles d'addition et de multiplication des séquences sont: $(a_0, a_1, a_2..., a_n, ...)+(b_0, b_1, b_2..., b_n, ...) = (a_0 + b_0, a_1 + b_1, a_2 + b_2..., a_n + b_n, ...)$ et la multiplication par

$$\begin{aligned}(a_0, a_1, a_2..., a_n, ...) \cdot (b_0, b_1, b_2..., b_n, ...) &= (c_0, c_1, c_2..., c_n, ...)\\ \text{avec } c_n &= \sum_{i=0}^{\infty} a_ib_{n-i}.\end{aligned}$$

Le produit " $\cdot$ " est dit le produit de convolution ou bien produit de Cauchy. Notons aussi que le produit d'un scalaire λ par une série $(a_0, a_1, a_2..., a_n, ...)$ est la série $\lambda \times (a_0, a_1, a_2..., a_n, ...) = (\lambda a_0, \lambda a_1, \lambda a_2..., \lambda a_n, ...)$.
Pour un anneau unitaire A, on note par $A[[X]]$ ou bien $A^{\mathbb{N}}$ l'ensemble des seriés formelles à coefficients dans A.
Muni de ces deux opérations, l'ensemble des séries formelles est un espace vectoriel sur A avec $0_A = (0, 0, 0, ...)$ la série nulle pour l'opération addition, et $1_A = (1, 0, 0, ...)$ la série unité pour le produit de convolution. Le symbole X est la série $(0, 1, 0, 0.....)$. On a $X^2 = (0, 1, 0, 0.....) \cdot (0, 1, 0, 0.....) = (0, 0, 1, 0, 0.....)$, $X^3 = (0, 1, 0, 0.....) \cdot (0, 1, 0, 0.....) \cdot (0, 1, 0, 0.....) = (0, 0, 1, 0, 0.....) \cdot (0, 1, 0, 0.....) = (0, 0, 0, 1, 0, 0.....)$. Avec la convention que $X^0 = (1, 0, 0....)$. A cet effet, on a donc:

$(a_0, a_1, a_2..., a_n, ...) = a_0 \times (1, 0, 0....) + a_1 \times (0, 1, 0, 0.....) + a_2 \times (0, 0, 1, 0, 0.....) + ... + a_n \times (0, 0, 0, ..., 1, 0, 0.....) + ...$

$\longleftrightarrow\ a_0X^0 + a_1X + a_2X^2 + ... + a_nX^n + ... = \sum_{n\geq 0} a_nX^n.$

1. **Generating function**

The ordinary generating function for the sequence $(a_0, a_1, a_2, ...a_n, ...)$ is the power serie (série formelle):

$$G(x) = a_0 + a_1X + a_2X^2 + ... + a_nX^n + ... = \sum_{n\geq 0} a_nX^n$$

A generating function is a formal power series in the sense that we usually regard X as placeholder rather than a number (on s'intérèsse pas ici à la convergence de la série mais plutôt a une algèbre sur ces séries et aux problemes de dénombrements). We regard the algebraic expression $G(x) = a_0 + a_1X + a_2X^2 + ... + a_nX^n + ...$ as convient way to represent the entire sequence of numbers $(a_0, a_1, a_2, ...a_n, ...)$. Si la séquence $(a_0, a_1, a_2, ...a_n)$ est finie la série corréspondante n'est autre que le polynôme $G(x) = a_0 + a_1X + a_2X^2 + ... + a_nX^n$.

Exercice 1

Let $A(z) = 1 + z^2 + 3z^5$ and $B(z) = 4 + z + 2z^3 + z^5$. a) Compute $A(z) + B(z)$. b) Compute $A(z)B(z)$.

Solution:

a) $A(z) + B(z) = 5 + z + z^2 + 2z^3 + 4z^5$.

b) $A(z)B(z) = (1 \times 4) + (1 \times z) + \left(1 + 2z^3\right) + \left(1 \times z^5\right) + \left(z^2 \times 4\right) + \left(z^2 \times z\right) + \left(z^2 \times 2z^3\right) + \left(z^2 \times z^5\right) + \left(3z^5 \times 4\right) + \left(3z^5 \times z^5\right)$

$= 4 + z + 4z^2 + 3z^3 + 15z^5 + 3z^6 + z^7 + 6z^8 + 3z^{10}$.

Exercice1

Consider the following product:

$$(1 + X + X^2 + X^3 + ...) \cdot (1 + X^2 + X^4 + X^6 + ...)$$

What is the coefficient of X^7 in this product? What is the coefficient of X^n?

Solution

When the terms in the first bracket are multiplied by those in the second bracket we obtain just 4 terms of degree seven, which are: $X \cdot X^6$, $X^3 \cdot X^4$, $X^5 \cdot X^2$ and $X^7 \cdot 1$.

These four terms correspond to the four different ways of writing the number 7 as the sum of two nonnegative integers, where the second is a multiple of 2.

En d'autres mots, c'est le nombre de solutions de l'équation en nombres entiers positifs ou nuls e_1, e_2 suivante:

$e_1 + e_2 = 7$ avec les conditions $e_1 \geq 0$ et e_2 entier pair.

En général, c'est le nombre d'exposants e_1, e_2 tel que $X^{e_1} \cdot X^{e_2} = X^{e_1+e_2} = X^n$, c.à.d le nombre de solutions de l'équation en nombres entiers positifs ou nuls e_1, e_2 tel que: $e_1 + e_2 = n$ avec les conditions $e_1 \geq 0$ et e_2 un entier pair. Evidemment, c'est le coefficient a_n du terme a_nX^n obtenu après avoir effectuer le produit $(1 + X + X^2 + X^3 + ...) \cdot (1 + X^2 + X^4 + X^6 + ...)$ (a_n=les différentes façons d'obtenir X^n dans ce produit).

Exercice 2 Find the coefficient of X^n in the series $(1 + X + X^2 + X^3 + ...)^2$. Also, find the coefficient of X^n in the series $(1 + X + X^2 + X^3 + ...)^3$.

solutionThe terms involving X^n in the product $(1 + X + X^2 + X^3 + ...)^2$ are the

$(n + 1)$ terms: $1 \times X^n$, $X \times X^{n-1}$, $X^2 \times X^{n-2}$, ..., $X^n \times 1$. Then the coefficient of X^n is $n + 1$.

Examples of generating functions

On désigne par une double flèche $\longleftrightarrow$ la correspondance entre la séquence et la fonction génératrice correspondante.

a) $(0, 0, 0, 0, ...) \longleftrightarrow 0 + 0x + 0x^2 + 0x^3 + ... = 0$

b) $(1,0,0,0,.....) \longleftrightarrow 1+0x+0x^2+0x^3+....=1$
c) $(3,2,1,0,.....) \longleftrightarrow 3+2x+1x^2+0x^3+....=3+2x+x^2$
d) $(1,1,1,1,.....) \longleftrightarrow 1+x+1x^2+1x^3+...+1x^n+....$
$=\sum_{n=0}^{\infty} x^n = \lim_{n\longrightarrow\infty}\left(\sum_{k=0}^{n} x^k\right) = \lim_{n\longrightarrow\infty}\left(\frac{1-x^{n+1}}{1-x}\right) = \frac{1}{1-x}$ if $|x|<1$ (recall that this is the sum of an infinite geometric series $\sum_{n=0}^{\infty} x^n$).
On peut retrouver ce resultat sans avoir recours à la notion de limite :
On a $\left(1+x+x^2+x^3+...+x^n+...\right)(1-x)$
$=\left(1+x+x^2+x^3+...+x^n+...\right)-\left(x+x^2+x^3+...+x^n+...\right)=1$
and then $\left(1+x+x^2+x^3+...+x^n+...\right)=\frac{1}{1-x}$.
e) $(1,1,1,1,.....) \longleftrightarrow 1+x+1x^2+1x^3+...+1x^n+....=\frac{1}{1-x}$,
f) $(1,-1,1,-1,.....) \longleftrightarrow 1-x+x^2-x^3+x^4...+....=\frac{1}{1+x}$,
g) $\left(1,a,a^2,a^3,.....\right) \longleftrightarrow 1+ax+a^2x^2+a^3x^3+........=\frac{1}{1-ax}$,
h) $(1,0,1,0,.....) \longleftrightarrow 1+x^2+x^4+x^6+........=\frac{1}{1-x^2}$,

2. Operations on Generating functions

2.1 Scalling (multiplication par un scalaire c)

If $(f_0,f_1,f_2.....) \longleftrightarrow F(x)$ then $c(f_0,f_1,f_2.....)=(cf_0,cf_1,cf_2.....) \longleftrightarrow cF(x)$.
$(cf_0,cf_1,cf_2.....) \longleftrightarrow cf_0+cf_1x+cf_2x^2+... = c\left(f_0+f_1x+f_2x^2+...\right) = cF(x)$.

Example: $(1,0,1,0,1,0,....) \longleftrightarrow 1+x^2+x^4+x^6+...=\frac{1}{1-x^2}$
Multiplying the generating function by 2 gives
$\frac{2}{1-x^2}=2\left(\frac{1}{1-x^2}\right)=2\left(1+x^2+x^4+x^6+........\right) \longleftrightarrow 2(1,0,1,0,1,0,.....)$.

2.2 Addition

Addition rule
If $(f_0,f_1,f_2.....) \longleftrightarrow F(x)$
and $(g_0,g_1,g_2.....) \longleftrightarrow G(x)$
then $(f_0+g_0,\ f_1+g_1,\ f_2+g_2,....) \longleftrightarrow F(x)+G(x)$.
In fact, $(f_0+g_0,\ f_1+g_1,\ f_2+g_2,....) \longleftrightarrow \sum_{n=0}^{\infty}(f_n+g_n)x^n$
$=\sum_{n=0}^{\infty} f_nx^n+\sum_{n=0}^{\infty} g_nx^n = F(x)+G(x)$

Example

$(1,1,1,1,1,1,...) \longleftrightarrow \frac{1}{1-x}$
$+\ (1,-1,1,-1,1,-1.....) \longleftrightarrow \frac{1}{1+x}$
————————————————————————————————————
$=(2,0,2,0,2,0,...) \longleftrightarrow \frac{1}{1-x}+\frac{1}{1+x}=\frac{(1+x)+(1-x)}{(1+x)(1-x)}=\frac{2}{1-x^2}$

2.3 Right shifting (décalage à droite)

If $(f_0,f_1,f_2.....) \longleftrightarrow F(x)$, then $\Big(0,0,...0,\ f_0,f_1,f_2.....\Big)$ (shift k positions=décalage de k positions)$\longleftrightarrow x^kF(x)$.

The idea of this rule is $\Big(0,0,...0,\ f_0,f_1,f_2.....\Big) \longleftrightarrow x^k f_0 + x^{k+1} f_1 + x^{k+2} f_2 + ... = x^k \left(f_0 + f_1 \times x + f_2 \times x^2 + ...\right) = x^k F(x)$.

Example : le décalage de la séquence $(1,1,1,1,1,1,...)$ de k positions donne la séquence

$(0,0,...,0,1,1,1,1,1,1,...) \longleftrightarrow x^k + x^{k+1} + x^{k+2} + ... = x^k \left(1 + x + x^2 + ...\right) = \frac{x^k}{1-x}$.

2.4 Differentiation

(derivative rule). If $(f_0,f_1,f_2.....) \longleftrightarrow F(x)$ then $(f_1,2f_2,3f_3,...) \longleftrightarrow F'(x)$.

The idea behind this rule is $(f_1,2f_2,3f_3,...) \longleftrightarrow f_1 + 2f_2 x + 3f_3 x^2 + ... = \frac{d}{dx}\left(f_0 + f_1 x + f_2 x^2 + f_3 x^3 + ...\right) = \frac{d}{dx} F(x)$.

For example, try to find the generating function of sequences of the squares $(0,1,4,9,16,.....)$.

The procedure is to begin with the generating function for $(1,1,1,1,.....)$.

$(1,1,1,1,.....) \longleftrightarrow \frac{1}{1-x}$

$(1,2,3,4,.....) \longleftrightarrow \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$,

$(0,1,2,3,4,.....) \longleftrightarrow x.\frac{1}{(1-x)^2} = \frac{x}{(1-x)^2}$,

$(1,4,9,16,...) \longleftrightarrow \frac{d}{dx}\frac{x}{(1-x)^2} = \frac{1+x}{(1-x)^3}$,

$(0,1,4,9,16,...) \longleftrightarrow x \times \frac{1+x}{(1-x)^3} = \frac{x(1+x)}{(1-x)^3}$. Thus, the generating function for squares is $\frac{x(1+x)}{(1-x)^3}$.

2.5 Products

product rule.

If $(a_0,a_1,a_2...,a_n,...) \longleftrightarrow A(x)$ and $(b_0,b_1,b_2...,b_n,...) \longleftrightarrow B(x)$

Then $(c_0,c_1,c_2...,c_n,...) \longleftrightarrow C(x) = A(x).B(x)$ where $c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + ... + a_n b_0$.

Exercice

What is the coefficient of the term x^{23} in the serie $\left(1 + x^5 + x^9\right)^{100}$?

Response

The only way to obtain x^{23} is to multiply $x^5 x^9 x^9 = x^{23}$. Then the coefficient is the number of selecting two nine and one five in the product $\left(1 + x^5 + x^9\right) \times \left(1 + x^5 + x^9\right) \times .... \times \left(1 + x^5 + x^9\right) = \left(1 + x^5 + x^9\right)^{100}$. The coefficient of x^{23} is $\begin{pmatrix} 100 \\ 2 \end{pmatrix} \times \begin{pmatrix} 98 \\ 1 \end{pmatrix} =$ le nombre de façons de choisir 2 termes x^9 et un terme x^5 dans le produit $\left(1 + x^5 + x^9\right)^{100}$.

Closed Formula

A closed formula for a sequence $(a_n)_{n\in\mathbb{N}}$ is a formula for a_n using a fixed finite number of operations on n. Just like defining a function in terms of n.

Examples

Here are some closed formulas for sequences:

- $a_n = n^2$ is the closed formula for the sequence : $0,1,4,9,16,25,36,...$
- $a_n = \frac{n(n+1)}{2}$ is the closed formula for the sequence : $0,1,3,6,10,15,...$

Recursive definition (or inductive definition) for a sequence $(a_n)_{n\in\mathbb{N}}$ consists of a recurrence relation: an equation relating a term of the sequence to previous terms (terms with smaller index) and an initial condition: a list of a few terms of the sequences (one less than the number of terms in the recurrence relation)

Examples

Here are a few recursive definitions for sequences:

- $a_n = 2a_{n-1}$ with $a_0 = 1$.
- $a_n = 2a_{n-1}$ with $a_0 = 27$.
- $a_n = a_{n-1} + a_{n-2}$ with $a_0 = 0$ and $a_1 = 1$.

Example

Find a_6 in the sequence defined by $a_n = 2a_{n-1} - a_{n-2}$ with $a_0 = 3$ and $a_1 = 4$.

Solution. We know that $a_6 = 2a_5 - a_4$. So to find a_6, we need to find a_5 and a_4. We have $a_5 = 2a_4 - a_3$ and $a_4 = 2a_3 - a_2$. In the same manner $a_3 = 2a_2 - a_1$ and $a_2 = 2a_1 - a_0$. Thus

$a_0 = 3$,
$a_1 = 4$,
$a_2 = 2 \times 4 - 3 = 5$,
$a_3 = 2 \times 5 - 4 = 6$,
$a_4 = 2 \times 6 - 5 = 7$,
$a_5 = 2 \times 7 - 6 = 8$,
$a_6 = 2 \times 8 - 7 = 9$.

We can guess a closed formula for the $n-th$ term of the sequence is $a_n = n+3$. To be sure we plug this in the formula of the recurrence relation:

$2a_{n-1} - a_{n-2} = 2\left((n-1)+3\right) - \left((n-2)+3\right)$
$= 2n + 4 - n - 1 = n + 3 = a_n$. Since $a_0 = 0 + 3$ and $a_1 = 1 + 3 = 4$ are the correct initial conditions, we conclude we have the correct closed formula.

Finding closed formulas, or recursive definitions, for sequences is not trivial. There is no one method for doing this. Just like in evaluating integrals or solving differential equations, it is useful to have a bag of tricks you can apply. One useful method is to relate a given sequence to another sequence for which we already know the closed formula.

Example

Use the formula $T_n = \frac{n(n+1)}{2}$ and $a_n = 2^n$ to find closed formulas for the following sequences.

1. $(b_n) : 1, 2, 4, 7, 11, 16, 22, ...$
2. $(c_n) : 3, 5, 9, 17, 33, ...$
3. $(d_n) : 0, 2, 6, 12, 20, 30, 42, ...$
4. $(e_n) : 3, 6, 10, 15, 21, 28, ...$
5. $(f_n) : 0, 1, 3, 7, 15, 31, ...$
6. $(g_n) : 3, 6, 12, 24, 48, ...$
7. $(h_n) : 6, 10, 18, 34, 66, ...$
8. $(j_n) : 15, 33, 57, 87, 123, ...$

Solution

The first few terms of $(T_n)_{n\geq 0}$ are $0, 1, 3, 6, 10, 15, 21, ...$(these are called the triangular numbers). The first terms of $(a_n)_{n\geq 0}$ are $1, 2, 4, 8, 16, ...$Let's try to find formulas fo the given sequences:

1. $(b_n) : 1, 2, 4, 7, 11, 16, 22, ...$ Note that if we subtract 1 from each term, we get the sequence (T_n). So, $b_n = T_n+1$ and the closed formula is $b_n = \frac{n(n+1)}{2}+1$.

2. $(c_n) : 3, 5, 9, 17, 33, ...$Each term in this sequence is one more than a power of 2, so we might guess the closed formula is $c_n = a_n+1 = 2^n+1, ...c_n = a_{n+1}+1$ giving $c_n = 2^{n+1} + 1$.

3. $(d_n) : 0, 2, 6, 12, 20, 30, 42, ...$ This gives $\frac{d_n}{2} = T_n$, so this sequence has closed formula $d_n = n(n+1)$.

4. $(e_n) : 3, 6, 10, 15, 21, 28, .., e_n = \frac{(n+2)(n+3)}{2} ......e_n = T_{n+2}$.

5. $(f_n) : 0, 1, 3, 7, 15, 31, ... \Longrightarrow f_n = 2^n - 1$.

6. $(g_n) : 3, 6, 12, 24, 48, ... \Longrightarrow g_n = 3 \times 2^n$.

7.$(h_n) : 6, 10, 18, 34, 66, ... \Longrightarrow h_n = 2(2^n + 1)$.

8. $(j_n) : 15, 33, 57, 87, 123, .. \Longrightarrow j_n = 3((n + 2)(n + 3) - 1)$.

Example

Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 4$.

Solution. The characteristic polynomial is $x^2 - 6x + 9 = 0 \iff (x - 3)^2 = 0$. The solution to the recurrence relation has the form $a_n = a \times 3^n + b \times n \times 3^n$ for some constants a and b. The use of the initial conditions gives:

$a_0 = 1 = a \times 3^0 + b \times 0 \times 3^0 = a$

$a_1 = 4 = a \times 3 + b \times 1 \times 3 = 3a + 3b$. Since $a = 1$, we find that $b = \frac{1}{3}$. Therefore the solution to the recurrence relation is $a_n = 3^n + \frac{1}{3} \times n \times 3^n$.

Recall

Arithmetic sequences:

If the terms of a sequence differ by a constant, we say the sequence is arithmetic. Then the recursive definition is $a_n = a_{n-1} + d$ with $a_0 = a$. The closed formula is $a_n = a + d \times n$.

Geometric sequences

A sequence is called geometric if the ratio between successive terms is constant. If the initial term a_0 is a and the common ratio is r, then the recursive definition is $a_n = ra_{n-1}$ with $a_0 = a$ and the closed formula is $a_n = a \times r^n$.

Since we know how to compute the sum of the first n terms of arithmetic and geometric sequences, we can compute the closed formulas for sequences which have an arithmetic (or geometric) sequence of differences between terms.

Counting with Generating Functions

Generating functions are useful for solving counting problems. In problems involving choosing items from a set lead to generating functions by considering the coefficent of x^n be the number of ways to choose n items.

The generating function for binomial coefficients

$$\left\langle \binom{n}{0}, \binom{n}{1}, \binom{n}{2}, ..., \binom{n}{n}, 0, 0, 0, ... \right\rangle$$

$$\longleftrightarrow \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + ... + \binom{n}{n} x^n = (1 + x)^n$$

The coefficient of x^k in $(1+x)^n$ is $\begin{pmatrix} n \\ k \end{pmatrix}$ the number of ways to choose k distinct items from a set of size n.

Building generating functions that count

In first, suppose that we have to choose from a single element set $\{a_1\}$. The generating function for the number of ways to select k element from this set is simply $1+x$: we have 1 way to select zero elements set, and one way to selecte one element set, and 0 ways to select more than one element. Similary, the number of ways to select k elements from the set $\{a_2\}$ is given by the generating function $1+x$.

The convolution rule :

The generating function for choosing elements from a union of disjoint sets is product of the generating functions for choosing from each set. If $A(x)$ is the generating function for selecting items from the set A and $B(x)$ is generating function for selecting items from the set B then the generating function for selecting items from the union $A \cup B$ is $A(x)B(x)$.

Example: The generating function for the number of ways to select from the set $\{a_1, a_2\}$ is:

$$\underset{\text{generat funct for selecting } a_1}{(1+x)} \times \underset{\text{generat funct for selecting } a_2}{(1+x)} = \underset{\text{generat funct for selecting from } \{a_1,a_2\}}{(1+x)^2} =$$

$1+2x+x^2$.

We have 1 way to select zero elements, 2 ways to select one element, 1 ways to select two elements and 0 ways to select more than two elements.

Repeated applications of this rule gives the generating function for selecting from the set $\{a_1, a_2, ..., a_k\}$,

$$\underset{\text{generat funct for selecting } a_1}{(1+x)} \times \underset{\text{generat funct for selecting } a_2}{(1+x)} ... \times \underset{\text{generat funct for selecting } a_k}{(1+x)} = \underset{\text{generat funct for selecting from } \{a_1,a_2,...,a_k\}}{(1+x)^k}$$

Choosing items with repetition (choix avec répétition)

Let's consider the combination of n objects with repetition taken r at time, we denote this by R_r^n or K_r^n,we think of it as having n kinds(types) of objects with an unlimited supply (réserve illimitée) of each, from which we wish to select r objects (nous pensons qu'on a n types d'objets, avec une quantité illimitée de chacun d'entre eux, parmi lesquels nous souhaitons sélectionner r objets avec répétition).

Just as before

the choice of x_1 with repetition is :

$(1 + x_1 + x_1x_1 + x_1x_1x_1 + ...)$

Similary, the choice of x_i with repetition is $1 + x_i + x_ix_i + x_ix_ix_i + ...$

Then the generating function of the choices with repetition is:

$(1 + x_1 + x_1x_1 + x_1x_1x_1 + ...)(1 + x_2 + x_2x_2 + x_2x_2x_2 + ...) ... (1 + x_n + x_nx_n + x_nx_nx_n + ...)$, since we are interested in the selections that include exactly r of $x's$ and we don't distinguish among the $x's$ we let $x_1 = x_2 = ... = x_n = x$, and we get then $\left(1 + x + x^2 + x^3 + ...\right)^n$. Each term that select exactly r of $x's$ contributes 1 to the coefficient of x^r. We have $\left(1+x+x^2+x^3+...\right)^n = \left(\frac{1}{1-x}\right)^n = 1+...+ (R_r^n)\,x^r+....$

Generalisation of binomial coefficients

Newton defined $\begin{pmatrix} \alpha \\ r \end{pmatrix}$ for non integer α as $\begin{pmatrix} \alpha \\ r \end{pmatrix} = \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-2)...(\alpha-r+1)}{1.2.3...r}$ and claimed that, if $|x|<1$ then

$(1+x)^\alpha = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} x^k$, for all α. Using this result, we find $(1-x)^{-n} = \sum_{k=0}^{\infty} \begin{pmatrix} -n \\ k \end{pmatrix} (-x)^k$. Since (R_r^n) is the coefficient of x^r in this sum, we find $(R_r^n) = \begin{pmatrix} -n \\ r \end{pmatrix} (-1)^r = (-1)^r \frac{(-n)(-n-1)(-n-2)...(-n-r+1)}{1.2.3...r} (-1)^r = \frac{(n)(n+1)(n+2)...(n+r-1)}{1.2.3...r} = \begin{pmatrix} n+r-1 \\ r \end{pmatrix}$ (combination with repetition).

Another way of proof

Suppose we have n kinds of objects $x_1, x_2, ..., x_n$. We select r objects, and set them down in order of increasing subscripts with separation point:

$$x_1x_1x_1 \cdot x_2x_2x_2x_2 \cdot x_3 \cdot \cdot \cdot \cdot x_nx_n$$

Even if we have no x_i for some i, we will include the seperation point

$$\cdot \cdot x_3 \cdot \cdot x_5x_5 \cdot \cdot \cdot \cdot x_{n-1}x_{n-1} \cdot$$

thus we always have r objects plus $(n-1)$ separation points. Once we choose the positions of the separation points, we have completely determined the set being selected

$\Longrightarrow$ Thus there are as many subsets of size r (with repetition) as there are ways of selecting $n-1$ separation points from among $r+n-1$ possible positions, this number is $\begin{pmatrix} r+n-1 \\ n-1 \end{pmatrix}$.